1

Number Sense

Number concepts and skills form the core of elementary school mathematics. Thus, a great deal of instructional time is devoted to topics related to quantity. One of the goals of instruction in arithmetic is for students to become numerically powerful and proficient. There are a number of components that are involved in numerical proficiency: conceptual understanding, procedural proficiency, strategic competence, adaptive reasoning, and productive disposition (Kilpatrick, Swafford, and Findell 2001). Being numerically proficient goes far beyond being able to compute accurately and efficiently; it entails understanding and using various meanings, relationships, properties, and procedures associated with number concepts and operations. It also involves the capacity to explain and justify one's actions on numbers and to use strategies appropriately and efficiently. How do we promote the development of children's numerical power? Certainly the instructional tasks we set for students have a bearing on what students learn. Equally important, however, is what we ourselves understand about numbers and number systems, for this knowledge contributes to our ability to ask students questions and provide learning experiences that are mathematically significant.

1. Classifying Numbers

We use numbers in many ways, for many purposes. Numbers can represent quantities—13 apples, 6 eggs. They can indicate relative position—the 5th person in line. They can define measurements—the distance between two towns, the temperature on a thermometer. Numbers can also be identifiers; we think of a telephone number or the number on a football jersey as "labels."

In addition to these everyday uses of numbers, we classify numbers in sets. For example, consider the set of whole numbers—0, 1, 2, 3, 4, and so on. When we add or multiply whole numbers, the answer is always a whole number. But what happens when we subtract whole numbers? Certainly problems such as $32 - 14$ and $5 - 3$ result in whole number answers. But we run into trouble when we try to record a drop in the outside temperature to below 0° or calculate $3 - 8$, because the answer isn't in the set of whole numbers. We need a broader set of numbers that includes negative numbers. Similarly, solving problems like $7 \div 2$ or $10 \div 3$ calls for a set of numbers that includes fractions. Defining different sets of numbers in mathematics becomes more important as our questions about how much and how many become more complex.

A third way to classify numbers is to examine particular characteristics of numbers. Within the set of whole numbers, for example, some numbers are even and some are odd. This characteristic is obvious to us as adults but not so clear to young children. Some numbers are divisible by 2, some by 3, and some by both 2 and 3. There are triangular numbers and square numbers, prime numbers and composite numbers. When the set of numbers includes decimals, there are some decimals that can be written as fractions and some that do not have a fractional representation. Classifying numbers by characteristics helps us make generalizations about sets of numbers and can provide insights into the relationships among numbers. For example, the number 36 is even so it is divisible by 2; we can arrange 36 beans in pairs. It is also a composite number, divisible by 1, 2, 3, 4, 6, 9, 12, 18, and 36. It is a square number, which means that 36 objects can be arranged in a rectangular array to make a square. Some of the characteristics of 36 are related—its being even and its being divisible by two. Other characteristics of 36 are not connected—its being even has nothing to do with its being a square number. Figuring out the why and why not of these relationships leads to a deeper understanding of specific numbers as well as of sets of numbers.

Classifying Numbers by Use

There are six different ways that we use numbers: for rote counting (saying numbers in sequence), for rational counting (to count objects), as cardinal numbers (to name "how many" objects are in a set), as ordinal numbers (to name the relative positions of objects in sets), for measurements (to name "how much" when we measure), and as nominal or nonnumeric numbers (for identification).

When numbers are verbally recited without referring to objects, they are being used in sequence. This is often referred to as *rote counting*. When we state that the number 5 comes after 4 and before 6, we are referring to numbers as part of a sequence. Adults sometimes consider rote counting a simple mechanical task. However, for very young children, counting in sequence is not at all simple—at least not at the outset. Children must learn the names of the numbers in the proper order and make sense of the patterns. In addition, a child's ability to count aloud well does not mean that he or she can correctly count a set of objects. Later, when the quantity meaning of numbers has been established, rote counting can be used to find answers to computations such as 16 + 3: by counting on from 16, saying 17, 18, 19. Number in the context of sequence can also be used to compare quantities—some children use counting to determine whether 25 or 52 is the larger quantity, knowing that the larger number will be said after the smaller number.

In *rational counting*, objects or events are matched with a number name. For example, when someone asks, *How many candies did you eat?*, you point to the last candy in the box and say, *This will be number 5*. It is through rational counting and matching number names and objects one-to-one that children start to understand the concept of quantity. Young children benefit from repetitively counting objects such as pennies, blocks, stones, acorns, and beans—they not only are learning the number names in sequence but also associating those numbers with specific amounts. Repetition appears necessary for children to develop effective strategies for counting items once and only once and to invent advanced strategies such as grouping. Many teachers introduce a weekly counting activity in which every student in the class counts and records the number of objects in a set. When the quantity is large (two- and three-digit amounts), students can be asked to group the objects and then to count the groups.

A *cardinal number* describes how many there are in a set; it can be thought of as a quantifier. In the statement *I ate 5 candies*, the number 5 is used to tell how many candies were eaten. Children gradually grasp the idea that the last number stated in a rational counting sequence names the amount in that set, but you can help students understand this meaning by using labels in a cardinal context (*I have 4 dogs*, not *I have 4*). When children consistently use numbers to tell "how many," even if the quantity is inaccurate, they are displaying an understanding of cardinality.

An *ordinal number* indicates the relative position of an object in an ordered set. Most children learn the initial ordinal numbers—*1st*, *2nd*, and *3rd*—in the context of ordinary situations. They construct the others from sequence, counting, and cardinal contexts. Understanding ordinal numbers is based on counting but lags behind: 95 percent of five-year-old children can say the first eight number words, but only 57 percent can say the first five ordinal names; only about 25 percent of entering kindergarten students can point to the third ball, but about 80 percent can create a set of three objects (Payne and Huinker 1993). Pointing to objects while using the word number (*number 1*, *number 2*, *number 3*) and linking these names with the ordinal numbers (*1st*, *2nd*, *3rd*) are techniques for helping students learn the ordinal numbers.

Number as a *measure* is used to indicate the size, capacity, or amount obtained by measuring something—how many units along some continuous dimension are being considered. We use number as a measure when we say that a kitchen has 14 feet of counter space. (Did you visualize one counter 14 feet long or 14 separate feet of counter space?) The idea of a continuous quantity—one in which the units run together and fractional values make sense—is difficult for students, since most of their other number experiences have involved whole numbers and counting individual items. Measurement tasks give you a chance to discuss different ways to represent a quantity (e.g., as a length or as separate objects) and what kinds of numbers are useful for describing how much instead of how many (e.g., dividing 17 feet of rope to make two jump ropes, each $8\frac{1}{2}$ feet long, makes sense; dividing 17 cars into two equal groups of $8\frac{1}{2}$ cars does not).

Nominal or nonnumeric numbers, such as telephone numbers, numbers on license plates, house numbers, bus numbers, numbers on athletes' jerseys, and social security numbers, are used for identification. Confusion arises for children because nominal numbers rarely correspond with other number meanings. For example, house number 5 does not have to be the fifth house on the street nor do there even have to be five houses on the street.

Classifying Numbers by Sets

Historically, the development of number systems closely followed the development of how numbers were used and manipulated. As people asked and then solved more and more sophisticated problems, they had to expand their ideas about number. For example, the first numbers used were the *counting numbers* (1, 2, 3, 4, 5, . . .), also known as the *natural numbers*. Some of the earliest written records of ancient cultures include numerical symbols in reference to counting people and animals. The development of trade and agriculture brought with it new needs for numbers: civilizations needed to be able to divide such things as land or goods. More than four thousand years ago, both the Egyptians and the Babylonians developed the idea of fractional portions so that they could represent quantities less than one in division.

While zero was used as a place holder by the Babylonians, the concept of zero as a number was not documented until the seventh century when it was credited to Hindu mathematicians. However, the work of an Arabian mathematician, Al-Khowarizmi, around A.D. 825, who also used zero as a number, was of greater importance to the development of mathematics in Western Europe. The translation of Al-Khowarizmi's work in the thirteenth century contributed to our use today of the Hindu-Arabic numerals. While the counting numbers as a set do not include zero, both the *whole numbers* (0, 1, 2, 3, 4, . . .) and the rational numbers do.

Europeans began to use negative numbers during the Renaissance, expanding the set of whole numbers to the set of integers, although the concept of having less than zero was used by both the Hindus and Arabs centuries earlier. The widespread use of negative and positive numbers came about because of increased commerce and the need to keep records of both gains and losses. Today, upper elementary or middle school students learn about *integers*—a set of numbers that includes zero, the counting numbers, and their opposites (the opposite of a number is the number that is the same distance from zero on the number line, only in the opposite direction). For example, . . . $^-5$, $^-4$, $^-3$, $^-2$, $^-1$, 0, 1, 2, 3, 4, 5, . . . are all integers. There are also negative fractions and decimals, but they are not classified as integers because they are not opposites of natural (counting) numbers.

Throughout history, different sets of numbers have been defined and then used to answer questions of how much and how many. In school, students must learn about and use different types of numbers to answer these same questions. In the early grades, children deal primarily with whole numbers. However, elementary-age children also use numbers from other sets. Children who live in cold climates are more likely to be introduced to integers at an early age, since the weather is an important topic of conversation. Temperatures of 5 below zero ($^-5$) or a wind chill of $^-20$ have meaning for these children!

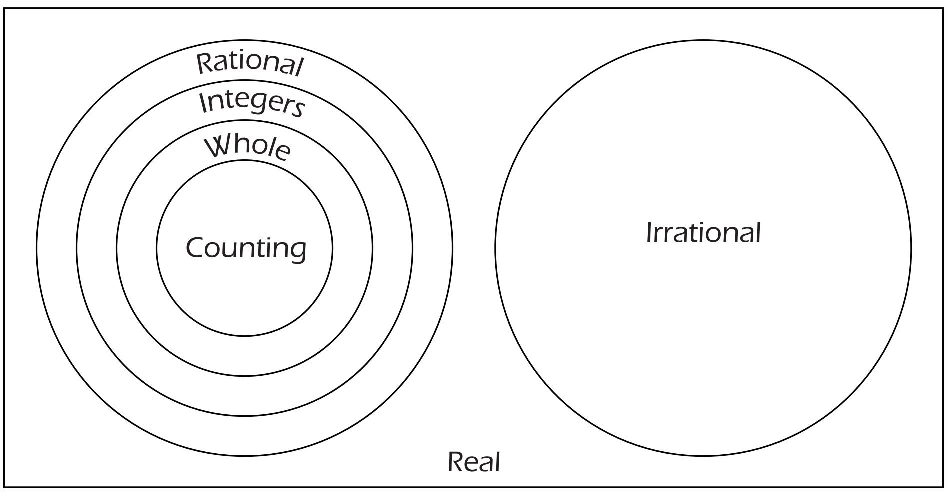
When students learn about fractions, they have to deal with a new set of numbers—*rational numbers*. Rational numbers enable us to answer questions about how many and how much using whole amounts and parts of whole amounts. Numbers that can be expressed as a ratio of two integers are *ratio*nal numbers, hence the name. More formally, all rational numbers can be written in the form $\frac{a}{b}$ (a ratio) where *a* and *b* represent integers and *b* does not equal zero. The set of rational numbers includes whole numbers and negative numbers. The rational number set also includes all fractions and some of the decimal numbers that are between integers, $^{-}26\frac{1}{3}$, $^{-}4.35$, $\frac{7}{8}$, and 1.75, for example.

Why are some decimal numbers rational numbers? Many decimals can be represented as a fraction ($0.5 = \frac{1}{2}$ and $3.7 = 3\frac{7}{10}$). All repeating decimals such as 0.333 . . . and 0.277 . . . are classified as rational numbers; they also can be represented as fractions ($\frac{1}{3}$ and $\frac{5}{18}$, respectively). Not all decimals, however, are rational numbers. Many decimals cannot be written as a ratio of two integers because they have decimal expansions that do not terminate or become periodic (repeat in some way). These are known as the *irrationals* (not ratios, hence the name). Many irrational numbers are found by taking the roots of rational numbers. The root of a number, N, is a number that can be multiplied by itself a given number of times to produce N. For example, the square root of 25 ($\sqrt{25}$) is 5 since $5 \times 5 = 25$. The cube root of 8 ($\sqrt[3]{8}$) is 2 since $2 \times 2 \times 2 = 8$. Both of these numbers are rational. However, the square root of 2 is 1.414213562 . . . and this number is irrational. The square root of other numbers that are not perfect squares, such as $\sqrt{3}$, $\sqrt{11}$, and $\sqrt{24}$, are also irrational. The decimal expansion of these numbers cannot be written as a fraction. But taking the roots of rational numbers is not the only way to find an irrational number. There are an infinite number of irrational numbers that are not roots, such as π, e, and numbers like 0.34334333433334333334 . . . . The irrational and rational number sets are infinite sets but, surprisingly, the irrational set has more elements.

Together the sets of rational and irrational numbers form the set of *real numbers*. Real numbers are used in all applications: measuring, comparing, counting, or determining quantities. Not all numbers in mathematics are real numbers, however ($\sqrt{^{-}7}$, for example, isn't a real number). These "imaginary" numbers belong to the set of complex numbers and are studied in high school and college.

It may help to visualize the different sets of numbers using a Venn diagram:

Sets of Numbers



Notice how most of these sets of numbers nest within each other. As numerical questions and answers become more complicated, students move from using “inside” sets of numbers to the full range of numbers in the real number system.

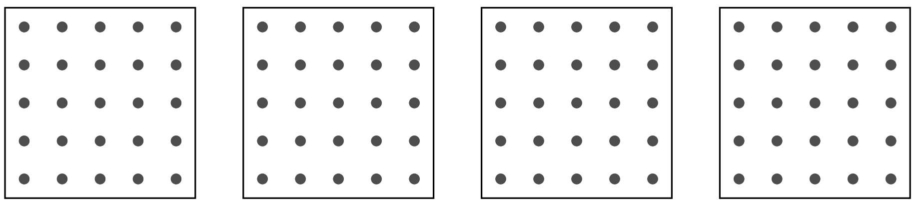
Activity

Exploring Irrational Numbers

Objective: explore lengths of line segments that are classified as irrational numbers.

If irrational numbers represent decimal expansions that do not terminate and do not repeat, can they be used to indicate measurements?

1. Use geoboards or sheets of dot paper similar to those in the picture below and construct or draw fourteen line segments of different lengths with dots as endpoints.
2. Determine the lengths of the line segments in units without using a measuring tool.
3. Which of the lengths represent rational numbers and which of the lengths represent irrational numbers? Explain how you know.
4. Without using the square root key on your calculator, use your calculator to determine the approximate length of one of the irrational number line segments to three decimal places.

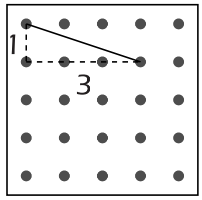


Things to Think About

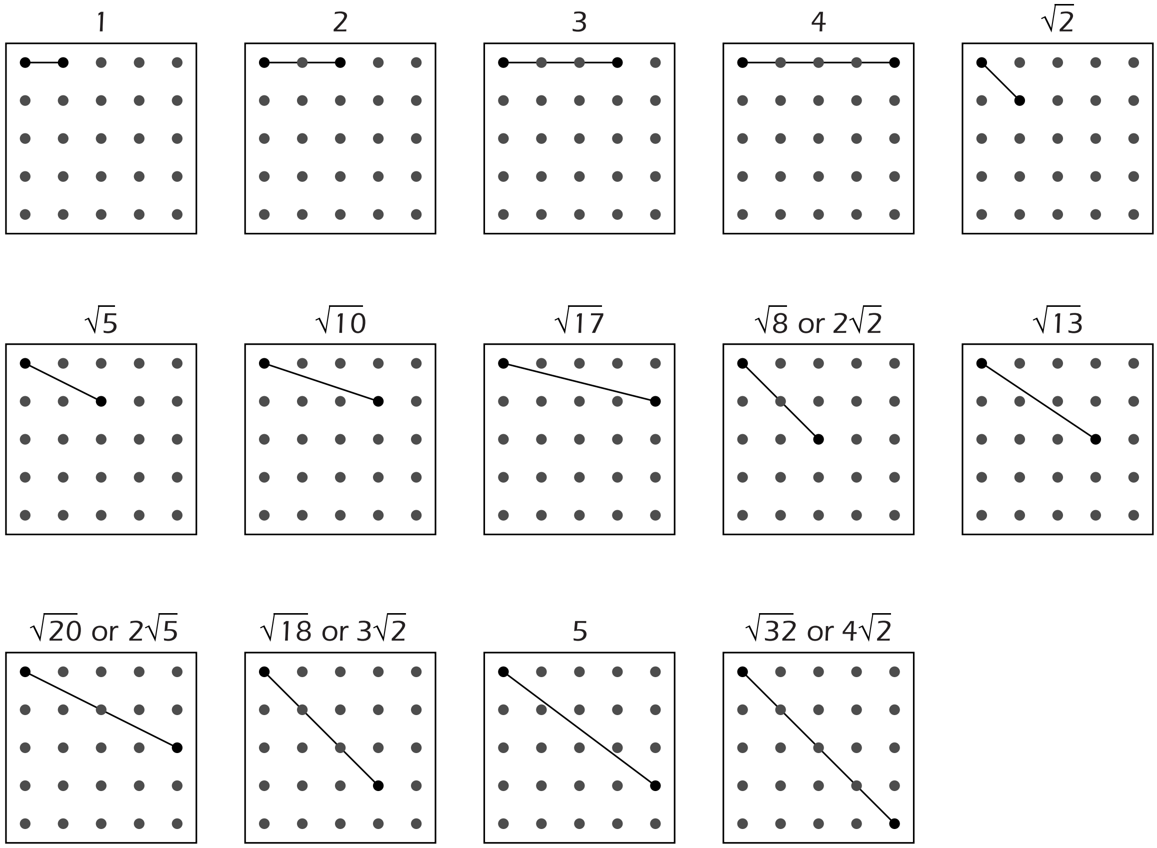
In order to find all fourteen line segments of different lengths on the geoboard, consider starting at one point and systematically connecting it to other points horizontally, vertically, or diagonally. Make sure each line segment is a unique length and not a duplicate of previously drawn segments. The orientation or placement of the line segment is not of interest in this activity, simply the length of the segment.

How do we determine the length of a diagonal line segment? One way is to apply the Pythagorean Theorem. First, draw a right triangle so that the diagonal segment is the hypotenuse of the triangle. (The hypotenuse is the side of a right triangle opposite the right angle.) The legs or sides of the triangle will be perpendicular to each other. The Pythagorean Theorem ($a^2 + b^2 = c^2$) enables us to find the length of the hypotenuse of a right triangle, given the lengths of the legs of the triangle. In the equation, a and b represent the lengths of the legs and c represents the length of the hypotenuse. Next, substitute values into the formula $a^2 + b^2 = c^2$. For example, the line segment shown on page 7 forms a right triangle with legs 1 unit and 3 units long. Applying the Pythagorean Theorem, we can determine that the value of c is $\pm\sqrt{10}$:

$$\begin{aligned} a^2 + b^2 &= c^2 \\ 1^2 + 3^2 &= c^2 \\ 1 + 9 &= c^2 \\ 10 &= c^2 \\ \pm\sqrt{10} &= c \end{aligned}$$



So what is the length of this segment? Every positive number has two square roots, a positive and a negative one. The symbol for the positive square root is $\sqrt{\ }$. Using a calculator, we find that $\sqrt{10}$ is approximately 3.16. The negative square root of 10 is shown by placing the negative sign outside the square root symbol ($^{-}\sqrt{10}$). The value of $^{-}\sqrt{10}$ is about $^{-}3.16$. Lengths of line segments are positive values, so the negative square root does not make sense in this situation. The length of this segment can be recorded as either $\sqrt{10}$ or about 3.16 units.



Five of the fourteen lengths can be classified as rational numbers (1, 2, 3, 4, and 5 units). These lengths can be represented as fractions or put another way as the ratio of two integers (can you determine what these ratios are?). The other nine line segments represent lengths that are classified as irrational numbers ($\sqrt{2}$, $\sqrt{5}$, $\sqrt{8}$, $\sqrt{10}$, $\sqrt{13}$, $\sqrt{17}$, $\sqrt{18}$, $\sqrt{20}$, $\sqrt{32}$ units). Notice that all of the irrational numbers in this problem are square roots. Irrational numbers can represent the measure of distances.

Every irrational number can be represented by a nonrepeating decimal expansion. Since the expansion continues forever, if we represent a length such as $\sqrt{10}$ units using the radical (root) sign, the answer is exact. But if we represent the length as a decimal, such as 3.162 units, the answer is approximate. Why? Because 3.162 is an estimate of the square root of 10. If you multiply 3.162 times itself (3.162×3.162), the product is close to 10 but not 10 exactly (9.998244). We can obtain a closer estimate of the square root of 10 by using a calculator or a computer to refine our estimate ($3.162277 \times 3.162277 = 9.999995824729$), but no matter what number we try, we will never find a number that when multiplied by itself has a product of exactly 10. This fact is what makes the square root of 10 an irrational number. By leaving the length as $\sqrt{10}$ units, we are giving an exact answer even though we cannot physically measure it exactly.

So how do we estimate the value of a square root without using the calculator's square root function? Let's use $\sqrt{5}$ as an example. First, think about numbers that when multiplied by themselves are close in value to 5. The numbers 2 and 3 ($2 \times 2 = 4$ and $3 \times 3 = 9$) fit the bill; we use these to estimate that the square root of 5 will be between them. It will be closer to 2 than 3 since 5 is much closer to 4 than 9. If we try 2.2 as the estimate, we find it is too small: $2.2 \times 2.2 = 4.84$. The number 2.3 is too large since $2.3 \times 2.3 = 5.29$. Using this information, make another estimate to two decimal places—perhaps 2.25. This estimate is too large, as is 2.24, but 2.23 is too small. Continuing with this guess-and-check strategy eventually leads us to 2.236 as an approximation to three decimal places for the square root of 5.

Another method for approximating the square root of a number dates back to the Babylonians and is often referred to as the "divide and average" method. Let's again use $\sqrt{5}$. Use what you know about multiplication to find a number that when multiplied by itself is close to 5 (2 or 3). Make a first estimate for the square root that is between these two numbers (say, 2.2). Using a calculator, divide 5 by 2.2 ($5 \div 2.2 = 2.27272727$). Now average the original estimate (also known as the divisor) and quotient [$(2.2 + 2.272727273) \div 2 = 2.23636363$]. Use this average as the new estimate and repeat the process. In other words:

$5 \div 2.2 = 2.27272727$	divide by estimate
$(2.2 + 2.27272727) \div 2 = 2.23636363$	average estimate and quotient
$5 \div 2.23636363 = 2.23577236$	divide by new estimate
$(2.23636363 + 2.23577236) \div 2 = 2.236067995$	average new estimate and quotient

Since the square root of 5 using the square root function on the calculator is 2.236067977, this method works quite well. Any ideas why?

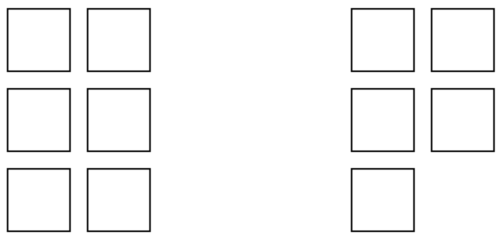
You may wonder whether eventually, by trying numbers with more and more decimal places, you'll ever find a number that when multiplied by itself produces exactly 5. The answer is no, but the proof of this is beyond the scope of this book. Elementary and middle school students might enjoy examining the decimal expansion of a particular irrational such as $\sqrt{5}$, which they can find on the Web, to see for themselves that it does not repeat or terminate in the given number of decimal places. While an example does not prove that the irrational number's decimal expansion does not terminate or repeat, it still can be a powerful way to help students understand the complexity of this set of numbers. ▲

Classifying Numbers by Particular Characteristics

Certain characteristics are specific to certain sets of numbers. For example, whole numbers can be separated into even and odd numbers, but that categorization doesn't make sense when considering rational numbers, because all rational numbers can be divided by 2 and result in a rational answer. With whole numbers, only those that can be divided by 2 and result in another whole number are even. Likewise, fractions can be classified into one of two groups depending on whether their decimal representation terminates or repeats, but that classification doesn't have any relevance when thinking about counting numbers.

Classifying numbers by certain characteristics helps identify number patterns (e.g., the last digit in an odd number is either 1, 3, 5, 7, or 9) and leads to generalizations about numbers. The more students know about a set of numbers, the more powerful they are in making sense of numerical situations. Since a great deal of time in the elementary grades is spent using whole numbers, this section focuses on some of the characteristics of those sets of numbers.

An important characteristic of numbers is whether they are even or odd. (We often think only of whole numbers as being even or odd, but technically negative numbers may also be classified this way.) There are a number of ways for students to check whether a number is even or odd. One is to take counters corresponding to the quantity designated by the number (e.g., 12) and try to divide them into two equal groups. If there is one extra counter left over, the number is odd; if not, the number is even. Taking counters and putting them into two equal groups is equivalent to dividing a number by 2. If the result has no remainder when divided by 2, the number is even. Using multiplication to represent this relationship, even numbers can be written in the form $2 \times n$ where n is an integer ($12 = 2 \times 6$ since $12 \div 2 = 6$; $184 = 2 \times 92$ since $184 \div 2 = 92$). But if the result when divided by 2 has a leftover or remainder, then the number is odd. Again using multiplication to represent this relationship, odd numbers can be written in the form $(2 \times n) + 1$. For example, $11 \div 2 = 5$ r 1 and $11 = (2 \times 5) + 1$, and $185 \div 2 = 92$ r 1 and $185 = (2 \times 92) + 1$. A third way to explain evenness is by using the analogy of dancing partners—when you have an even number of people everyone has a partner or can form pairs. If even and odd numbers are represented using objects, the idea of partners can be shown concretely:



Knowing that the ones digit of all even numbers is either 0, 2, 4, 6, or 8 lets students quickly identify even numbers, but the pattern in itself does not help students understand the concept of evenness. They need many experiences in which they either separate counters into two equal groups or form "partners" with them and then decide whether the number quantity the counters represent is even or odd.

Is the number 0 even or odd? This confuses students since if they have 0 counters, how are they to check if the counters can be separated into two equal groups? They can't so we must use another method for checking: divide by 2 and see if the result has a remainder. When we divide 0 by 2, the quotient is 0 with no remainder ($0 \div 2 = 0$). Furthermore, we can represent 0 as ($2 \times n$) where $n = 0$ ($2 \times 0 = 0$). Notice that there is no value for n that works to make ($2 \times n$) + 1 equal to 0. Thus, 0 is an even number. Most second graders can simply be told that 0 is even, but older students (and some precocious second graders) should be asked to make sense of why.

Activity

2

Even and Odd Numbers, Part 1

Objective: investigate sums of even and odd numbers.

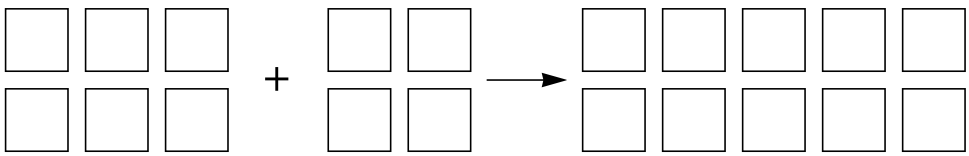
Consider the following questions:

1. Is the sum of two even numbers even or odd?
2. Is the sum of two odd numbers even or odd?
3. Is the sum of one odd number and one even number even or odd?

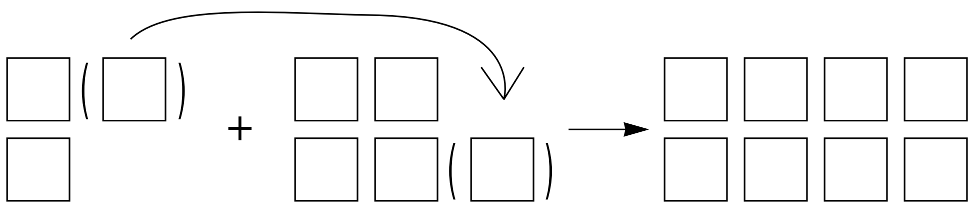
Using words, drawings, symbols, or concrete materials, explain why these types of sums occur.

Things to Think About

Why is the sum of two even numbers an even number? If a number is even, it can be represented using "partners," or pairs. Combining two numbers that are each separable into groups of two results in a third number (the sum) that is also separable into groups of two. Another way of stating the relationship is that the sum can be divided into two equal groups:



The sum of two odd numbers is also an even number. An odd number has what we can refer to as a "leftover"—when an odd number of counters is divided into two equal groups or into groups of two, there is always one extra counter. When we combine two odd numbers together, the "leftovers" from each odd number are joined and form a pair:



Using this same line of reasoning, it follows that the sum of one even and one odd number will always be an odd number. ▲

3

Even and Odd Numbers, Part 2

Objective: investigate products of even and odd numbers.

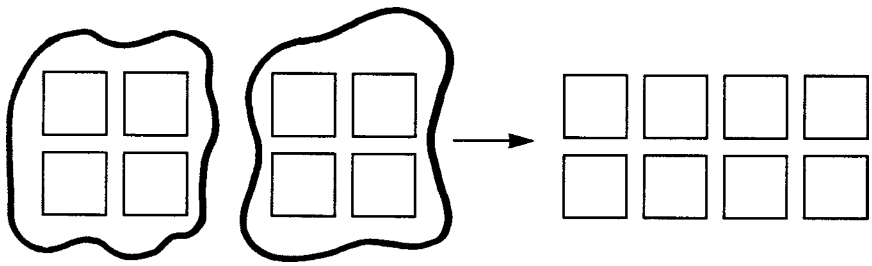
Consider the following questions:

1. Is the product of two even numbers even or odd?
2. Is the product of two odd numbers even or odd?
3. Is the product of one odd number and one even number even or odd?

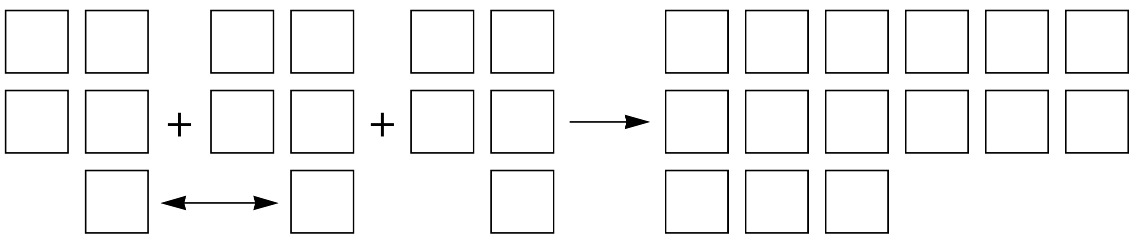
Using words, drawings, symbols, or concrete materials, explain why these types of products occur.

Things to Think About

Why is the product of two even numbers an even number? One way to interpret multiplication is as a grouping operation. For example, 2×4 can be interpreted as two groups of four. Since there are an even number in each group (4) and an even number of groups (2), the total amount is always going to be an even number:



By generating examples, you discovered that the product of two odd numbers is an odd number. Why? In this case, think of multiplication as repeated addition. Using 3×5 to illustrate, we can interpret this multiplication as $5 + 5 + 5$—and we notice we are adding an odd number an odd number of times. Since every two odd numbers when added together will total an even number, the extra odd number will remain without a partner and will affect the total by making it odd:



The third case, an even number multiplied by an odd number, always gives an even product. Both interpretations of multiplication, grouping and repeated addition, can be used to explain why. ▲

Math Matters: Understanding the Math You Teach, Grades K–8, Second Edition

Another important classification of numbers is based on factors. A factor of a number divides the number evenly—there is no remainder. The numbers 2, 3, 4, and 6 are factors of 24, but 5 and 7 are not. Mathematicians have been intrigued for centuries by prime numbers: a prime number has exactly two unique factors, 1 and itself. Composite numbers have more than two factors.

Activity

Rectangular Dimensions and Factors

Objective: use rectangular arrays to visualize the differences among prime numbers, composite numbers, and square numbers.

Materials: graph paper.

Draw on graph paper as many different-shape rectangles as possible made from 1 square, 2 squares, 3 squares, . . . 25 squares. What are the dimensions of your rectangles? When you use 6 squares there are four whole number dimensions possible—1 and 6, and 2 and 3—but when you use 7 squares there are only two whole number dimensions—1 and 7. The whole number dimensions of these rectangles are also called *factors* of these numbers. Use the dimensions of rectangles formed from each number to list the factors of the numbers from 1 through 25. Remember that a square is a special kind of rectangle. Sort the numbers based on the number of factors.

Things to Think About

Does the orientation of the rectangles matter? Is a 2-by-5 rectangle the same as a 5-by-2 rectangle? The rectangles are different in that they show two distinct arrays, but they cover the same area: a 2-by-5 rectangle can be rotated to represent a 5-by-2 rectangle. For this activity we are interested in the dimensions of the rectangles (or the factors of the numbers), not in their vertical or horizontal orientation. What is important to note is that 2 and 5 are both factors of 10.

What did you find out about the number of factors? The number 1 has one factor (itself) and forms one rectangle (a 1-by-1 square); it is classified by mathematicians as a special number and is neither prime nor composite. Many numbers have only two factors and make just one rectangle: 2, 3, 5, 7, 11, 13, 17, 19, and 23. These numbers are the prime numbers. Prime numbers are defined as having exactly two unique factors. All the other numbers are composite numbers and have more than two factors; composite numbers can be represented by at least two unique rectangular arrays.

NUMBER	FACTORS	NUMBER OF RECTANGLES	PRIME OR COMPOSITE
1	1	1	Neither
2	1,2	2	Prime
3	1,3	2	Prime
4	1,2,4	3	Composite
5	1,5	2	Prime
6	1,2,3,6	4	Composite
7	1,7	2	Prime

8	1,2,4,8	4	Composite
9	1,3,9	3	Composite
10	1,2,5,10	4	Composite
11	1,11	2	Prime
12	1,2,3,4,6,12	6	Composite
13	1,13	2	Prime
14	1,2,7,14	4	Composite
15	1,3,5,15	4	Composite
16	1,2,4,8,16	5	Composite
17	1,17	2	Prime
18	1,2,3,6,9,18	6	Composite
19	1,19	2	Prime
20	1,2,4,5,10,20	6	Composite
21	1,3,7,21	4	Composite
22	1,2,11,22	4	Composite
23	1,23	2	Prime
24	1,2,3,4,6,8,12,24	8	Composite
25	1,5,25	3	Composite

Examine the rectangles representing the numbers 1, 4, 9, 16, and 25. Did you notice that in each case one of the rectangles that can be formed is also a square? Ancient Greek mathematicians thought of number relationships in geometric terms and called numbers like this square numbers, because one of the rectangular arrays they can be represented by is a square. The square numbers have an odd number of factors, whereas the other numbers examined have an even number of factors. Numbers that are not square always have factor pairs. For 12, for example, the factor pairs are 1 and 12, 2 and 6, and 3 and 4. But square numbers always have one factor that has no partner other than itself. For 9, for example, 1 and 9 are a factor pair, but 3 is its own partner because $3 \times 3 = 9$. The factor of a square number that has no partner—3 for the square number 9—isn't listed twice. Therefore, the factors of 9 are 1, 3, and 9, and the factors of 16 are 1, 2, 4, 8, and 16—an odd number of factors.

The square numbers between 25 and 200 are 36, 49, 64, 81, 100, 121, 144, 169, and 196. The terminology *squaring a number*—eight squared (8^2), for example—comes from the fact that when you multiply 8 by 8, one way to represent this amount geometrically is in the form of a square array with eight unit squares on each side. ▲

One of the earliest mathematicians to study prime and composite numbers was Eratosthenes (Era-toss′-the-neez), a Greek. He is known for a method of identifying prime numbers, the Sieve of Eratosthenes. Since the time of Eratosthenes, the hunt for prime numbers has occupied many mathematicians. A French mathematician from the early Renaissance, cleric Marin Mersenne (1588–1640), proposed the following formula for generating prime numbers: $2^n - 1$, where n is a prime number. Substitute the numbers 1 through 7 for n in his formula. Which of your results were prime numbers? Did you find that when n was a prime number, another prime number was generated by the formula? If $n = 2$, then the Mersenne Prime is 3 ($2^2 - 1 = 3$), and if $n = 3$, a Mersenne Prime, 7, is created ($2^3 - 1 = 7$). For $2^5 - 1$, the Mersenne Prime is 31. The next Mersenne Prime is $2^7 - 1$, or 127, and there are several prime numbers between 31 and 127. Over the course of the last three hundred years, as mathematicians worked on finding primes using Mersenne's formula, they discovered that Mersenne's formula does not generate all possible prime numbers. But by 1947 it was determined that if $n = 2, 3, 5, 7, 13, 17, 19, 31, 61, 89, 107$, or 127, the formula does produce another prime number.

As counting numbers become larger, it becomes increasingly arduous to determine whether they are prime or not. Even high-speed computers have a difficult time examining extremely large numbers to determine whether they are prime. Therefore the current search for prime numbers examines only Mersenne Primes, because those numbers have the potential to be prime numbers. As of 2005, the largest prime number was a Mersenne Prime: $2^{254,964,951} - 1$. It contains over seven billion digits!

Other mathematicians fascinated by primes include Pierre de Fermat (1601–1665), Leonhard Euler (1707–1783), and Christian Goldbach (1690–1764). In a letter dated June 7, 1742, Goldbach presented his now famous conjecture to Euler, namely that any even number can be represented as the sum of two prime numbers. Test Goldbach's conjecture by finding sums for several even numbers using prime numbers as addends: $6 = 3 + 3$, $8 = 5 + 3$, $12 = 7 + 5$, $20 = 13 + 7$ are all examples that support Goldbach's conjecture. Students enjoy learning about prime numbers and investigating Goldbach's conjecture. (No one has ever found an even number greater than 2 that cannot be written as the sum of two prime numbers.)

The exploration of number characteristics is sometimes purely recreational. One interesting classification scheme involves labeling counting numbers as *perfect*, *abundant*, or *deficient*. These classifications are related to the factors of the numbers. A perfect number is equal to the sum of all its factors other than itself (e.g., 28 is perfect, since $28 = 1 + 2 + 4 + 7 + 14$). If the sum of the factors of a number (other than itself) is greater than the number, then it is classified as abundant (e.g., 18 is an abundant number, since $1 + 2 + 3 + 6 + 9 = 21$). Finally, a deficient number is one whose factors (other than itself) sum to less than the number (e.g., 16 is a deficient number, since $1 + 2 + 4 + 8 = 15$). Factors of a number other than itself are sometimes referred to as proper factors.

Activity 5

Perfect Numbers

Objective: learn about perfect, abundant, and deficient numbers.

Pick a number. List the proper factors. Find the sum of the proper factors, and then classify the number as perfect, abundant, or deficient. Pick another number, list the proper factors, find their sum, and classify the number. Can you find one number of each type?

Things to Think About

Were you able to find a perfect number? There aren't a lot of perfect numbers, only two that are smaller than 30: 6 and 28. Some abundant numbers are 12, 18, 20, 24, and 30. Examples of deficient numbers include 8, 9, 10, 14, 15, 21, and 25. All prime numbers are deficient numbers. Can you think of a reason that explains why? A prime number has only two factors, 1 and itself, so the only proper factor is 1. Therefore, the "sum" of the proper factors is also 1 and is always less than the number. Exposing students to a variety of classification schemes broadens their appreciation of the many roles numbers play in society. ▲

2. Understanding Numbers

Students' understanding of quantity is based on their understanding of number meaning and number relationships and develops over time. Young children use counting to make sense of quantity both by recognizing where numbers are in the counting sequence and by connecting counting and cardinality to actual objects. Their understanding is extended when they are able to consider sets of objects as parts of wholes and can conceptualize a number as comprising two or more parts. They eventually establish anchor quantities like 5 and 10 and realize that these groups can themselves be counted (e.g., 60 can be thought of as 6 groups of 10 not just 60 units). As students' understanding of number progresses, they establish relationships that link prime factors, factors, and multiples in numerous ways.

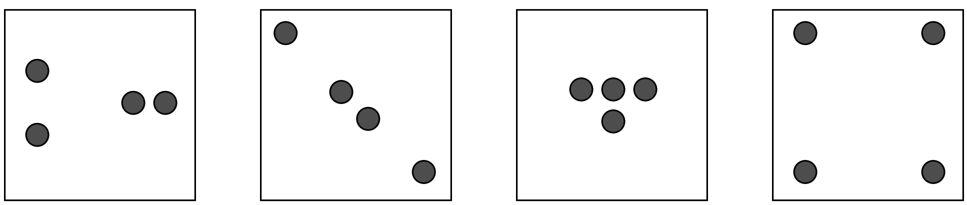
Counting is very important in developing students' early understanding of quantity. Rational counting progresses from objects that can be touched to objects that are only seen to the mental image of a group of objects. Learning to count by rote involves both memorization and identification of patterns. The first twelve numeral names are arbitrary and must be memorized. The numerals 13 through 19 follow a "teens" pattern—a digit name followed by *teen*, which represents a ten. Then the pattern shifts so that the tens value is presented first (twenty, thirty, . . .) followed by the ones digits. The transition at decades (e.g., counting on from 79 to 80 or on from 109 to 110) is especially difficult for children, in part because the decade names break the tens-ones pattern (we don't say twenty-ten, twenty-eleven) and require students to have learned the order of the decade names. Interestingly, the Chinese numerals for the decades are much more orderly and systematic—one-ten represents the number ten, two-ten is equivalent to twenty, three-ten represents thirty, and so on. Thus, 42 is read four-ten-two, and 89 is read eight-ten-nine.

Some children enter kindergarten being able to count to 10 or 20; this ability is strongly affected by opportunities to practice. Other children are unable to count when they come to school, because they have had limited or no practice. Furthermore, many early childhood programs don't include counting activities with quantities greater than 20. As a result it is very difficult for students to identify counting patterns, learn the decade names, and generalize how these patterns continue. Teachers need to increase the number of counting activities for children whose counting skills are weak so that they learn the number names in sequence. Asking students to discuss patterns in counting and to reflect on how numbers are said also promotes the development of these skills.

In order for students to attach quantitative meaning to number words, however, they must do a great deal of counting of things. This is sometimes referred to as "meaningful counting," "counting with understanding," or "rational counting." Young children work hard to master counting objects one by one and with time develop strategies for counting things accurately. Most mistakes occur because students don't count each member of the relevant set once and only once, perhaps because they are moving too many objects at a time or not attending to what they are doing. If you notice students who do not coordinate their verbal counting with their actions with regard to objects, you can show these students how to keep track of their counts by moving an object from the uncounted pile to the counted pile as you say each number (it helps to use a mat with a line down the center or a shoe box lid divided into two sections with string). In addition it helps children make generalizations about quantity if you vary the arrangement of objects to be counted (in a row, a circle, or a random pattern) and if you ask children to share their own strategies for keeping track of what they have counted (Ginsburg 1989). Counting pictures or things that can't be moved is a different skill. If possible, have children cross off each picture as it is counted.

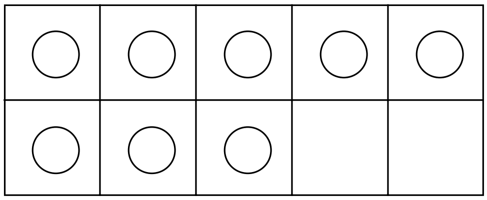
An important milestone occurs when children connect counting and cardinality—they understand that the last number counted indicates how many are in a set. Some children grasp the cardinal meaning of number as young as age two or three. When children start school without this understanding, teachers may be explicit: *The last number you say when counting tells how many objects you have. Watch me. One, two, three, four, five. Five. There are five marbles. Watch again.*

In addition to using counting to make sense of quantities, children also use imagery. In general, children are interested in developing more efficient ways to count, and over time they learn to "see" small numbers without counting. They instantly recognize ◆◆◆ as three. The identification of small quantities without counting is known as *subitizing* and appears to develop after children have had repeated practice in counting such sets. Sometimes students recognize only particular arrangements of objects such as the dots on a die. In order for students to extend their use of subitizing, it helps to vary the arrangement of the objects. For example, the number 4 can be shown using many different dot patterns:



Furthermore, when students identify without counting that there are four dots, they tend to think about this quantity as a group instead of as individual dots. Visualizing a set of objects as a "group" is an important step toward being able to decompose quantities into small groups (e.g., 5 can be thought of as a group of 2 and a group of 3). Dot cards, dominoes, and dice are often used in instruction because they present a variety of arrangements of dots (or pips) for small numbers.

The ten-frame (a 2-by-5 array of squares) is a tool used to help students organize visual patterns in terms of 5 and 10. As students count objects, they place each counted object into a cell of the ten-frame. It is important in terms of visualization that students first fill the top row of the ten-frame (5) and then move to the second row. This helps them see that a quantity such as 8 can be thought of as a group of 5 and a group of 3.



A ten-frame can be used in spatial relationship activities that focus on identifying groups of objects. For example, you can place eight pennies on a ten-frame and cover it with a sheet of paper. Quickly uncover the ten-frame and observe which students count and which students "see" the parts (5 and 3) and/or the whole group (8). Exposure to this type of activity will help students develop visual images of specific-size groups. The ten-frame also helps students visualize and quantify 10. In the ten-frame above, eight cells are filled and two cells are empty—leading us to consider 10 as 8 plus 2 and as 5 plus 3 plus 2.

Understanding Part-Whole Number Relationships

Recognizing small groups or quantities is a skill students use to develop more sophisticated understanding about number. A major milestone occurs in the early grades when students interpret number in terms of part and whole relationships. A part-whole understanding of number means that quantities are interpreted as being composed of other numbers. For example, the number 6 is both a whole amount and comprises smaller groups or parts such as 1 and 5 or 2 and 4. Research indicates that students instructed using a part-whole approach do significantly better with number concepts, problem solving, and place value than those students whose instruction focuses just on counting by ones. Students first learn about part-whole relationships for the numbers 0 through 10. Solid understanding of the many relationships inherent to number takes time. Therefore it is not unusual for some second graders to have limited part-whole constructs for the numbers 7 through 12.

There are many instructional activities that encourage part-whole thinking. For example, you can show students a tower of interlocking cubes and ask them to determine all the ways they can divide the tower into two parts. Be sure to ask students to link numerical symbols with the concrete manipulations; when dividing a set of cubes into two groups, students should record their findings on paper. After they finish, they should reconnect the blocks in order to revisualize the "whole" amount. Or give your students six objects, ask them to put some in one hand, the rest in their other hand, and to put their hands behind their backs. Then ask each child how many objects are in one hand (for example, 4), how many are in the other hand (2), and how many objects he or she had to start with (6). It may appear that these activities are teaching addition, and they do provide a strong conceptual base for addition and subtraction. However, the focus is actually on the meaning of the quantity in terms of composition/decomposition.

Activity 6

Exploring Part-Whole Relationships

Objective: explore patterns when numbers are decomposed additively into two parts.

A number can be broken into two parts in different ways. Eleven, for example, is made up of 4 and 7, among other combinations. Order matters, because we are looking at the number of combinations, not the actual addends: 4 and 7 is different from 7 and 4. Zero is an acceptable part. How many different two-part combinations of whole numbers are possible for 11? 7? 12? 35? n? What patterns do you notice in the combinations? What generalizations can you make?

Things to Think About

For 11, there are 12 two-part combinations; for 7, there are 8 two-part combinations; for 35, there are 36 two-part combinations; and for n, there are $n + 1$ two-part combinations. What patterns did you observe in the combinations? For the number 7, the combinations are 7 and 0, 6 and 1, 5 and 2, 4 and 3, 3 and 4, 2 and 5, 1 and 6, and 0 and 7; as one number decreases, the other number increases. The reason there are $n + 1$ two-part combinations becomes clear if you look at the actual combinations. Since zero (0) is a part, there are actually eight different numbers (0 through 7) that can be combined with other numbers: any of the eight numbers 0, 1, 2, 3, 4, 5, 6, or 7 can be the first part, and the corresponding quantity that creates 7 is the second part. ▲

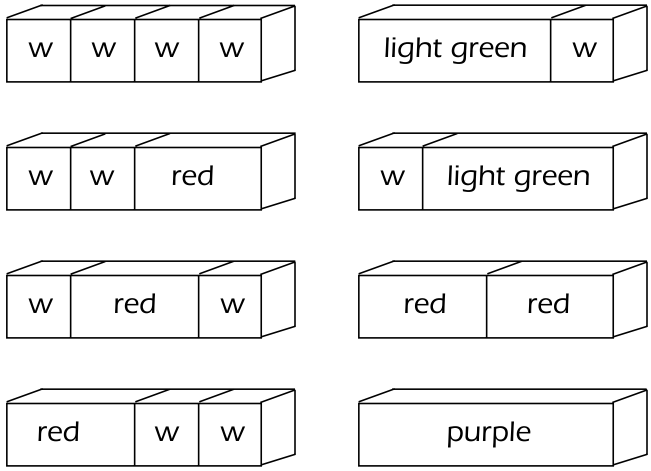
Activity 7

Exploring Combinations

Objective: explore patterns when numbers are decomposed additively into more than two parts.

Materials: Cuisenaire rods.

A purple Cuisenaire rod is 4 units long. There are eight different ways to combine other Cuisenaire rods to create a rod that is the same length as the purple rod:



You can use four rods (1-1-1-1), three rods (1-1-2; 1-2-1; 2-1-1), two rods (3-1; 1-3; 2-2), or one rod (4). The order of the parts matters: 1-1-2 is different from 1-2-1. How many different ways are there to create each of the other Cuisenaire rods? You may wish to list the different combinations in a table.

ROD (LENGTH)	PARTS THAT MAKE THE ROD	NUMBER OF COMBINATIONS
White (1)		
Red (2)		
Light Green (3)		
Purple (4)	1 + 1 + 1 + 1, 1 + 1 + 2, 1 + 2 + 1, 2 + 1 + 1, 3 + 1, 1 + 3, 2 + 2, 4	8
Yellow (5)		
Dark Green (6)		

Things to Think About

For the smallest rod, white, there is only one way to show that length—with a white rod. The length of the next rod, red, can be made using two white rods or one red rod. There are four combinations that equal the length of the light green rod—light green; red and white; white and red; and white, white, and white. There are eight combinations for the purple rod, sixteen combinations for the yellow rod, and thirty-two combinations for the dark green rod.

ROD (LENGTH)	PARTS THAT MAKE THE ROD	NUMBER OF COMBINATIONS	
White (1)	1	1	2^0
Red (2)	1 + 1, 2	2	2^1
Light Green (3)	1 + 1 + 1, 1 + 2, 2 + 1, 3	4	2^2

Continued

ROD (LENGTH)	PARTS THAT MAKE THE ROD	NUMBER OF COMBINATIONS	
Purple (4)	1 + 1 + 1 + 1, 1 + 1 + 2, 1 + 2 + 1, 2 + 1 + 1, 3 + 1, 1 + 3, 2 + 2, 4	8	2^3
Yellow (5)	1 + 1 + 1 + 1 + 1, 1 + 1 + 1 + 2, 1 + 1 + 2 + 1, 1 + 2 + 1 + 1, 2 + 1 + 1 + 1, 2 + 2 + 1, 1 + 2 + 2, 2 + 1 + 2, 3 + 1 + 1, 1 + 3 + 1, 1 + 1 + 3, 1 + 4, 4 + 1, 3 + 2, 2 + 3, 5	16	2^4
Dark Green (6)	1 + 1 + 1 + 1 + 1 + 1, 1 + 1 + 1 + 1 + 2, 1 + 1 + 1 + 2 + 1, 1 + 1 + 2 + 1 + 1, 1 + 2 + 1 + 1 + 1, 2 + 1 + 1 + 1 + 1, 1 + 1 + 2 + 2, 1 + 2 + 1 + 2, 1 + 2 + 2 + 1, 2 + 2 + 1 + 1, 2 + 1 + 1 + 2, 2 + 1 + 2 + 1, 1 + 1 + 1 + 3, 1 + 1 + 3 + 1, 1 + 3 + 1 + 1, 3 + 1 + 1 + 1, 1 + 2 + 3, 2 + 1 + 3, 2 + 3 + 1, 1 + 3 + 2, 3 + 1 + 2, 3 + 2 + 1, 1 + 1 + 4, 1 + 4 + 1, 4 + 1 + 1, 3 + 3, 4 + 2, 2 + 4, 1 + 5, 5 + 1, 2 + 2 + 2, 6	32	2^5

What patterns did you observe in the number of combinations? As the rods got longer, by one centimeter each time, the number of combinations doubled. These numbers, 1, 2, 4, 8, 16, 32, . . . , are referred to as powers of two, since each of the numbers can be represented as 2 to some power: $1 = 2^0$, $2 = 2^1$, $4 = 2^2$, $8 = 2^3$, $16 = 2^4$, and $32 = 2^5$. Another pattern can be seen in the exponents; each exponent is one less than the length of the rod. For example, the yellow rod is 5 cm long and there are 2^4, or 16, ways to make a rod this length. The patterns observed in the number of combinations can be used to generalize the relationship to rods of any length; the number of combinations for each rod is $2^{(n-1)}$, where n is the length of the rod in units. Thus, there are 2^6, or 64, ways to make a black rod; 2^7, or 128, ways to make a brown rod; 2^8, or 256, ways to make a blue rod; and 2^9, or 512, ways to make the 10-centimeter orange rod. Notice how quickly the number of combinations increases. ▲

Understanding Multiplicative Number Relationships

So far in this section we have examined number from a counting and a part-whole (or additive) perspective. As students' understanding of number expands and as they move through formal schooling, they are introduced to multiplicative number relationships. One multiplicative relationship involves *factors* and *products*. For example, because 3 times 8 equals 24, 3 and 8 are factors and 24 is a product. Factors that are prime numbers are called *prime factors*. Factors are also called *divisors*, since they divide the number evenly with a zero as the remainder. Knowledge of factors and divisors is used when applying the rules of divisibility, determining the least common denominator, and finding common multiples.

Integers greater than 1 can always be expressed as the product of prime factors in one and only one way. The number 12 can be factored using the following primes: $2 \times 2 \times 3$. Twelve is unique in its composition of prime factors. Thus, changing one of the prime factors (e.g., $2 \times 2 \times 5$) results in a different product and thus a different number (20). The relationship between a number and its prime factors is so important that there is a theorem about it, the Fundamental Theorem of Arithmetic, which states that every integer greater than 1 can be expressed as the product of a unique set of prime factors. The order of the factors does not matter. For example, $2 \times 3 \times 5$ is the same as $3 \times 5 \times 2$. Each set of these prime factors equals 30, regardless of which factor is listed first, second, or third.

Activity

8

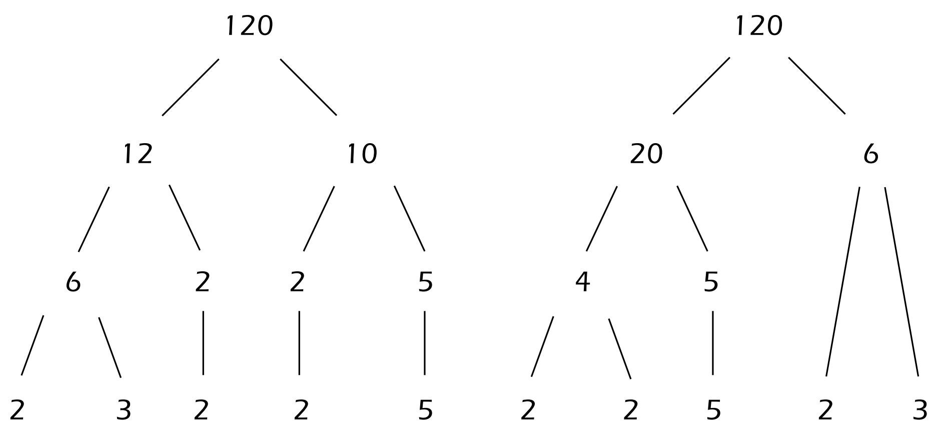
Prime Factors

Objective: understand how prime factors are combined multiplicatively to form numbers.

Determine the prime factors for 120. Then list all factors of 120. How can you use the prime factors of 120 to determine all the factors of 120? Is 120 divisible by 2, 3, 4, 5, 6, 7, or 8? How are the prime factors of 120 related to whether or not 120 is divisible by any of these numbers? Pick another composite number and determine its prime factors. What are the factors of your new number? What numbers divide evenly into your new number or, put in another way, are divisors of your new number?

Things to Think About

There are a number of ways to factor a number, but the most common approach is to use a factor tree. In a factor tree, the factors are represented on branches and the next-lower line of branches presents factors that produce the number above as a product. The tree continues until no number on the bottom line can be factored further. Here are two factor trees for 120. Notice that while the upper parts of the trees use different factors, the prime factors at the bottom of the trees ($2 \times 2 \times 2 \times 3 \times 5$) are the same:



How can the prime factors be used to determine all the factors of 120? First, it is important to remember that all the prime factors of 120 must be used; otherwise you will end up with a different number than 120. Using the commutative and associative properties, the prime factors of 120 can be combined in different ways to find all other factors. For example, multiplying two of the prime factors, 2 and 3, gives a product of 6, which is one factor. That leaves $2 \times 2 \times 5$, or 20, as the other factor that pairs with the 6. Some of the factors of 120 are:

$$4 \times 30 \text{ or } (2 \times 2) \times (2 \times 3 \times 5)$$
$$8 \times 15 \text{ or } (2 \times 2 \times 2) \times (3 \times 5)$$
$$10 \times 12 \text{ or } (2 \times 5) \times (2 \times 2 \times 3)$$
$$120 \times 1 \text{ or } (2 \times 2 \times 2 \times 3 \times 5) \times 1$$

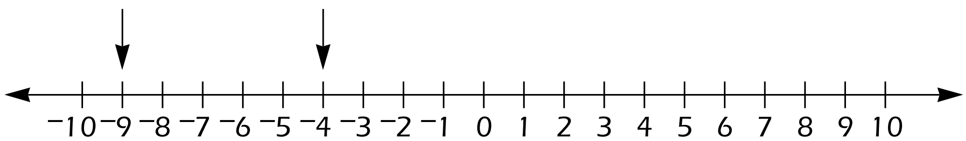
Continuing with this method, 16 factors of 120 can be found. They are 1, 2, 3, 4, 5, 6, 8, 10, 12, 15, 20, 24, 30, 40, 60, and 120.

The number 120 is divisible by 1, 2, 3, 4, 5, 6, 8, 10, 12, 15, 20, 24, 30, 40, 60, and 120, because each of these numbers is a factor of 120. Numbers that are not factors of 120, such as 7 and 9, do not divide evenly into 120. ▲

Understanding Negative and Positive Number Relationships

In the middle grades, students' experiences with numbers are expanded to include negative numbers. The topic is difficult for a number of reasons. Students not only must make sense of negative values with which many have little experience, but also are introduced to new symbols and vocabulary. Computations with negative and positive numbers are especially problematic, as the operations and algorithms are rarely understood and easily confused.

What are the important characteristics of this group of numbers? One is *direction*, which indicates whether a number is to the right of zero, and therefore positive, or to the left of zero, and therefore negative. We can use the concept of direction to compare numbers, since the farther a number is to the right on the number line, the greater it is. For example, which is greater, $^{-}4$ or $^{-}9$? Both are negative and are to the left of zero, but $^{-}4$ is greater than $^{-}9$ since $^{-}4$ is to the right of $^{-}9$ on the number line.



The other important characteristic when thinking about positive and negative numbers is the *magnitude* of the number. Magnitude is the distance of a number from 0. Five is 5 intervals from 0 and $^{-}2$ is 2 intervals from 0. We refer to the magnitude of a number as its absolute value. Put another way, the absolute value of a number is its distance from 0. This concept is so important that there is a particular symbol, $|n|$, to represent when we are referring to a number's magnitude or absolute value. The absolute values of both $^{-}6$ and 6 are the same ($|^{-}6| = 6$ and $|6| = 6$) since both $^{-}6$ and 6 are 6 intervals from 0, yet these numbers are in opposite directions from 0. Numbers that are the same distance from 0 but on opposite sides of 0 are known as *opposites*. What do we now know about the number $^{-}467$? It is a negative number and thus its direction is to the left of 0. It is less than 0 and is 467 intervals away from 0. It is a small number, because it is so far from 0 to the left. The opposite of $^{-}467$ is 467. Both numbers are the same distance from 0.

When we combine the two characteristics of direction and magnitude, we have powerful information to help us interpret, compare, and order all numbers, but in particular, negative numbers.

The language associated with comparisons is particularly important when working with negative numbers: we compare numbers using the terms *greater than*, *less than*, and *equal to*. Informal language used to compare quantities, such as *bigger*, *smaller*, and *larger*, can cause significant confusion when dealing with negative numbers. In the early elementary grades, when students work with positive numbers, there is general agreement on the meaning of comparative terms; students understand that if a number is "bigger" or "larger" than another, it is another positive number. Their knowledge comes from counting; the last number you say when counting is the greatest. But when students start working with negative numbers, informal language can be problematic. For example, which is larger, $^-8$ or $^-2$? Since when we count, 8 comes after 2, students often think $^-8$ is greater than $^-2$. Furthermore, the magnitude or distance from 0 to $^-8$ is greater than $^-2$, so again students think $^-8$ is greater. Also, a common model to help students think about negative numbers uses red chips for negative values and black chips for positive values. Eight red chips representing $^-8$ is physically more than two red chips representing $^-2$, even though $^-8$ is actually less than $^-2$.

One way to help students avoid these types of errors is to link the concepts of distance and magnitude to comparisons on the number line. Numbers to the right of other numbers are greater; 25 is greater than $^-47$ because it is to the right of $^-47$. Likewise, since $^-2$ is to the right of $^-8$ on the number line, $^-2$ is greater than $^-8$ (or we can say that $^-8$ is less than $^-2$). Which negative number is closer to 0 ($^-2$ is closer to 0 than $^-8$, for example) is another way of determining which of them is greater. However, which number is closest to 0 is not enough when comparing negative and positive numbers. For example, $^-1$ is closer to 0 than 5 but $^-1$ is not greater than 5; 5 is farther to the right on the number line. Students need to learn that the placement of a number on the number line is important when comparing the size of negative numbers, in particular. In addition, we should try to use only the proper terminology, *greater than* and *less than,* in order to eliminate any possible confusion about which characteristics we are considering. We want these comparison words to be correctly associated with the related concepts.

Positive numbers can be written in three ways: 35, $+35$ or $^+35$. A positive number does not have to have a plus sign; it is required only when there may be confusion as to the sign of the number. Sometimes the plus sign is raised, but it doesn't have to be. Negative numbers always have a negative sign in front of them (raised or not) and are often put in parentheses in number sentences where a subtraction sign is used and the negative sign is not raised (e.g., $5 - (-3)$) to help us differentiate between the negative sign and the subtraction sign. When using positive and negative numbers it is tempting to refer to them using the terms *plus* and *minus*. However, if we say *plus 20* instead of *positive 20* or *minus 7* instead of *negative 7*, it is easy to become confused: do we mean the positive and negative numbers or do we mean the operations of addition and subtraction?

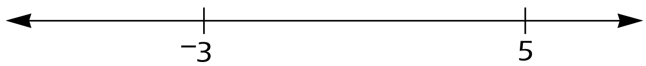
Two other terms are also applied to negative and positive numbers: *signed numbers* and *integers*. *Signed numbers* is an informal name for the set of negative and positive real numbers. It is usually used when we want to let others know we are talking about negative and positive numbers, including negative and positive fractions and decimals (and irrationals). *Integers*, on the other hand, are numbers that are formally defined as the whole numbers and their opposites—they do not include negative and positive fractions and decimals. Thus, some signed number are integers ($^{-}23$ and 149), but other signed numbers are not ($^{-}2.5$ and $13\frac{5}{8}$).

Activity

9 The Number Line

Objective: create a number line and generalize its important features.

1. Draw a blank number line (without numbers indicated). Make two marks anywhere along the line. Label the marks $^{-}3$ and 5. Where is 0? Where is 1?

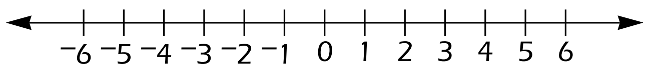


2. Place the following numbers on your number line: A) $^{-}2.3$; B) $^{-}\frac{1}{5}$; C) $\sqrt{23}$; D) $^{-}3\frac{3}{4}$; E) $|^{-}2|$; F) the opposite of $^{-}5\frac{1}{4}$; G) $^{-}\sqrt{13}$; H) x when $|x| = 4$.
3. Is 0 a positive number, a negative number, or neither? What is the opposite of 0? Explain your thinking.

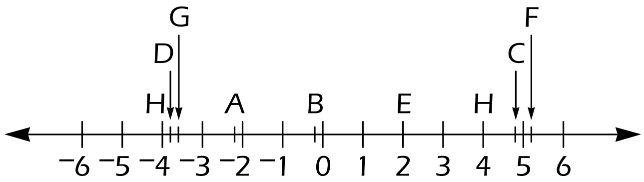
Things to Think About

The number line is an important representation that pictures numbers as points on a line. Every point on the real number line corresponds to exactly one real number, and every real number corresponds to exactly one point on the number line. Although we refer to a number as a point, this is not actually the case—a point really represents the magnitude of the number or its distance from zero.

What did you have to consider in order to construct your number line? First, determining the placement of 0 is essential, since numbers to the right of 0 represent positive values (values greater than 0) and numbers to the left of 0 represent negative values (or values less than 0). Second, the number line is symmetrical around 0 (namely, pairs of numbers are the same distance from 0). Thus both $^{+}5$ and $^{-}5$ are 5 intervals away from 0 and are 10 intervals apart. The idea that the line folds onto itself around 0 may be an image that helped you in numbering the line. Third, the intervals between consecutive integers must be the same but the actual length of the interval is arbitrary. There are 8 intervals between $^{-}3$ and 5, so we can make 7 equally spaced hash marks to represent the numbers $^{-}2$, $^{-}1$, 0, 1, 2, 3, and 4. The size of the intervals on your number line may be different from the one here because of where you originally placed $^{-}3$ and 5. Each interval represents one unit.



Constructing number lines and placing points on the line focuses our attention on both the size and direction of each number. Where do $\sqrt{23}$ and $^{-}\sqrt{13}$ belong? The square root symbol, often called a radical, indicates that the operation of evaluating the square root is to be performed. Thus the approximate values are 4.8 and $^{-}(3.6)$, respectively, which can then be placed on the number line. The absolute value symbol indicates that we must evaluate the absolute value of the number $|^{-}2|$, which is 2 (since it is 2 units from 0), and place 2 on the number line. The absolute value of a number is always positive since it is a distance (the number of intervals from 0). What is x when $|x| = 4$? In this case, x has two values. Both 4 and $^{-}4$ have an absolute value of 4.



Students always have questions about 0. Is it positive? Is it a number? Does it have an opposite? Zero is neither a positive number nor a negative number since there is no change in direction and it has no magnitude. The opposite of 0 is 0: numbers with the same absolute value are opposites, and opposites are numbers that are the same distance from 0 but on opposite sides of 0. ▲

3. Our Place Value Numeration System

In order to communicate ideas related to number, we must have a way of representing numbers symbolically. A numeration system is a collection of properties and symbols that results in a systematic way to write all numbers. We use the Hindu-Arabic numeration system, which was developed around A.D. 800.

One of the important features of the Hindu-Arabic numeration system is that it is a positional system: there are place values. Equally important is that it is based on repeated groupings of ten. For this reason the Hindu-Arabic system is also referred to as the *base ten* or *decimal numeration* system.

These two characteristics, place value and groupings by ten, require students to interpret numerals within numbers on two levels: place value and face value. The 5 in 58 has a place value representing the tens place; the face value of the 5 must therefore be interpreted to mean that five groups of ten, not five ones, are being considered. Sometimes students interpret numbers by considering only the face values of the digits. For example, students who have little understanding of place value might incorrectly assume that 58 and 85 represent the same quantity, since each numeral contains both a 5 and an 8. Likewise, when dealing with decimal fractions, students often state that 0.4 and 0.04 are equivalent if they consider only the face values (both numerals have a 4 in them) and not the place values.

Larger numbers are read by naming the period of each group of three digits. The three digits in each period represent the number of hundreds, tens, and ones making up that period. The periods to the left of hundreds are thousands, millions, billions, trillions, quadrillions, quintillions, and so on. Zillions, though used a great deal in literature, are not a mathematical period! The numeral 23,456,789 is read "twenty-three million, four hundred fifty-six thousand, seven hundred eighty-nine." It provides us with a great deal of information—there are 2 groups of ten million and 3 groups of one million (which is equivalent to 23 groups of one million); 4 groups of one hundred thousand, 5 groups of ten thousand, and 6 groups of one thousand (which is equivalent to 456 groups of one thousand); plus 7 hundreds, 8 tens, and 9 ones. While we say the quantity in each period as if we are only focusing on face value ("four hundred fifty-six thousand"), students must be able to interpret the place value meaning of each digit within the periods (which involves more than simply identifying the place value of each digit).

Helping Students Understand Our Place Value System

How can we help students understand our place value system? First, students must be able to organize objects into specific-size groups (tens, hundreds, thousands, . . .) and realize that they can count these groups. Many teachers use grouping activities to help their students think of quantities as groups of hundreds, tens, and ones. For example, students might count 147 beans by grouping the beans into sets of ten. Counting reveals that there are 14 groups of ten plus 7 additional beans. The notion that 147 can be represented by single units and by a variety of groups of tens and hundreds doesn't always makes sense to children; counting helps young students verify that the quantities, despite the different groupings, are equal. With sufficient experiences and guidance from a teacher, students internalize these relationships.

Second, students must apply their understanding of the part-whole relationship to partitioning numbers into groups based on powers of ten (e.g., hundreds, tens, and units). For example, 123 is equivalent to 1 group of a hundred plus 2 groups of ten plus 3 ones. Note that if students aren't quite sure that they can count groups or that these groupings are equivalent to a number of units, they will have difficulty decomposing a number into place values. Students' understanding of these notions is gradual and appears to develop first with two-digit numbers and then with three-digit numbers. Students often partition numbers into place values when they devise their own algorithms for adding and subtracting numbers, because this allows them to deal with the component parts separately.

Other ideas related to place value that students must make sense of involve the relationships among the groups. There is growing evidence from research and national assessments that many young students do not understand to any depth the multiplicative relationship among groups and thus have difficulty comprehending, for example, that there are 32 hundreds in 3,289 or that ten hundredths equals one tenth. Decomposing numbers into equivalent parts using place values other than the face values is another problematic area for students; many students can identify that 78 is equal to 7 tens and 8 ones but are not so sure that 78 can be represented with 6 tens and 18 ones, or 5 tens and 28 ones. The standard subtraction algorithm is an example of using equivalent representations—to subtract 38 from 62 using the standard algorithm we have to think of (or regroup) the 62 as 5 tens and 12 ones.

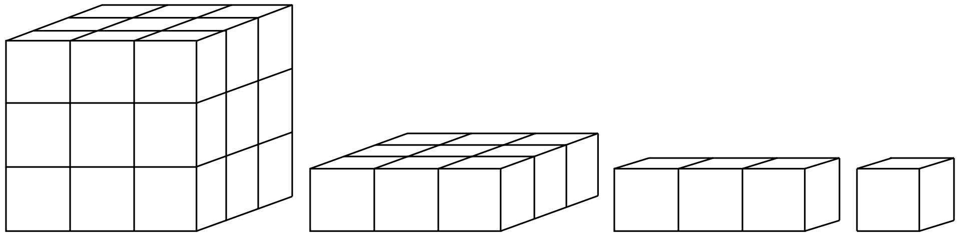
Activity

10

Analyzing a Different Numeration System

Objective: group and record quantities using a different number base system.

A factory packages chocolate truffles in cartons that hold 27 truffles, in trays that hold 9 truffles, in 3-packs, and in individual boxes:



When an order for truffles arrives at the factory, workers package the truffles in as few boxes as possible: that is, they never use three packaging units of the same size (e.g., three trays) but instead always repack the truffles into the next-larger unit (e.g., one carton). Thus, an order for 38 truffles would be filled using one carton (27), one tray (9), and two single boxes (2). The factory's method for recording this shipment is 1102_{truffles}: each digit corresponds to the number of cartons (1), trays (1), 3-packs (0), and singles (2).

You have just begun work at the truffle factory. Determine the packaging for orders of 10, 49, 56, 75, and 100 truffles. (You may want to make sketches of the packages.)

Now record shipments of 1 to 20 truffles. You start with 1, which you record as 1_{truffles}; 2 truffles you record as 2_{truffles}; 3 truffles would be packaged in a 3-pack and recorded as 10_{truffles}. Continue recording the packaging of 4 through 20 truffles.

Things to Think About

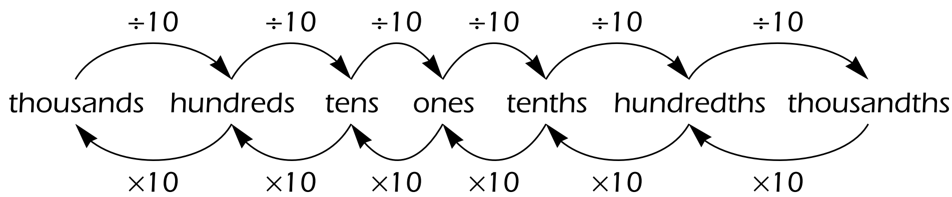
The record entry for 10 truffles is one tray and one single (101_{truffles}), since 9 + 1 = 10. The record entry for 49 truffles is one carton, two trays, one 3-pack, and one single (1211_{truffles}), for 56 truffles it is two cartons and two singles (2002_{truffles}), and for 75 truffles it is two cartons, two trays, one 3-pack, and zero singles (2210_{truffles}). Did you notice that the groupings in the truffle factory are based on threes? When there are three singles they are regrouped into a 3-pack, three 3-packs are regrouped into a tray, and three trays are regrouped into a carton. Each package is three times as large as the next smallest package. The truffle factory record system is similar to our numeration system in that it uses groupings, but it is dissimilar in that the groupings are based on threes rather than tens.

Was your record entry for 100 truffles 321_{truffles}? What happens when we have three cartons? Because the factory regroups whenever there are three of any-size package, we need to regroup and make a new package that is equivalent to three cartons and holds 81 truffles. Let's call it a crate! Thus, following the packaging rules, when packaging 100 truffles we use one crate (81), zero cartons, two trays (18), zero 3-packs, and one single—10201_{truffles}.

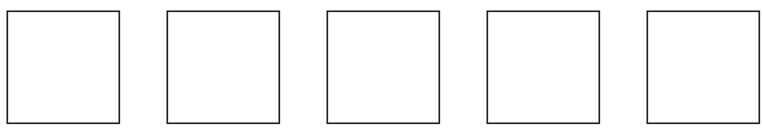
Packaging truffles in groups of three provides you with some insight into the difficulties students have making sense of grouping by tens. The chocolate factory uses a base three numeration system. To make sense of the symbols—21012_{truffles}, for example—you have to understand the place values of the digits and the grouping relationships among the place values associated with base three. Just as you had to clarify packaging by threes, students have to master base ten relationships. Pictures and models can help students make sense of our base ten system.

The record entries for packaging 1 through 20 truffles are 1, 2, 10, 11, 12, 20, 21, 22, 100, 101, 102, 110, 111, 112, 120, 121, 122, 200, 201, and 202. Notice that only the digits 0, 1, and 2 are used to represent all the packaging options. Whenever there are three of any package, the numeral three isn't used to represent this quantity because the truffles are repackaged (regrouped) into a larger package. That is, the number after 212 is not recorded as 213 but as 220, since the three singles are repackaged into another 3-pack. The packaging notation for the truffle factory is identical to counting in base three. Our base ten numeration system has ten digits—0, 1, 2, 3, 4, 5, 6, 7, 8, and 9. Whenever there are ten of any item, we regroup (repackage) the quantity and record the amount using the next place value. Thus, in base ten when we have 19 items and add one additional item, we don't record that we have one ten and ten ones but regroup to show that we have two tens (20). ▲

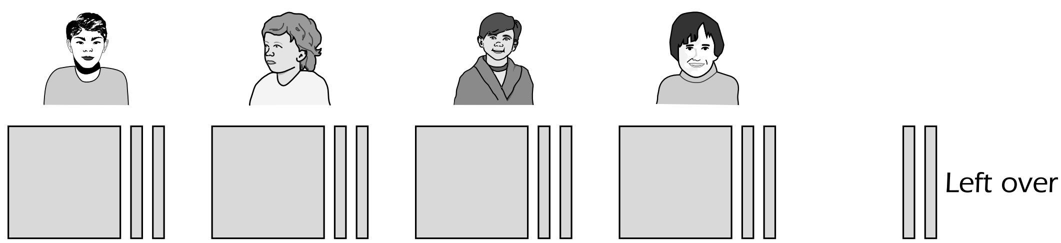
So far we have examined place value concepts using quantities greater than one. Grouping whether in base ten or base three enables us to represent large quantities efficiently and with only a few symbols. But what about when we have to represent quantities less than one? How do we apply the idea of groups of 10 with small quantities? When dividing up quantities we can partition groups into 10 equal pieces. In other words, in our numeration system we divide (or partition) groups by 10 when representing quantities less than one.



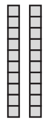
Let's start with 5 brownies that we want to share evenly with 4 people. How much will each person receive? We can represent the 5 brownies using squares.



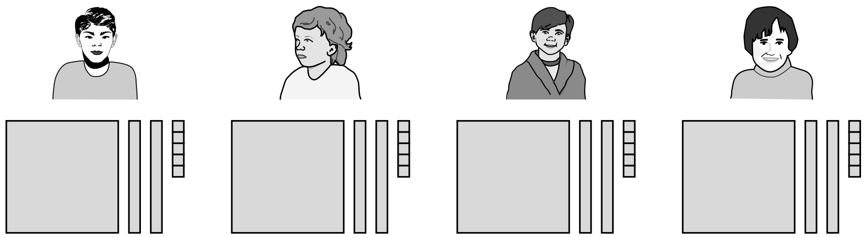
Each person will receive one full brownie and a part of a brownie. To determine the part of the one brownie that each person gets, we must follow our dividing rule—cut the extra brownie into 10 equal pieces called *tenths*. Distribute 2 of these tenths to each person, leaving 2 pieces left over.



Following the dividing rule again, we cut the 2 tenths into 10 pieces each. We now have 20 pieces, each one-hundredth of the original square. These 20 hundredths can be distributed evenly to the four people—each person receiving 5 hundredths.



Therefore, each person receives 1 whole brownie, and 2 tenths plus 5 hundredths of a brownie, for a total of 1.25 brownies.



Activity

Place Values Less than One

Objective: understand decimal place values.

Materials: graph paper cut into 10-by-10 squares.

Try the cutting activity yourself to help you better understand how decimal place values are created. For example, explore sharing 1 brownie equally among 6 people and 17 brownies equally among 8 people.

Things to Think About

In order to share 1 square (brownie) equally among 6 people, we have to cut the brownie. Following the dividing rule, we cut it into 10 pieces called tenths. Each of the 6 people gets one-tenth with 4 tenths left over. Divide each of the 4 pieces by 10 and we have 40 hundredths. These can be distributed to the 6 people, each receiving 6 hundredths (40 ÷ 6 = 6 with 4 left over). Take these 4 hundredths and cut each of them into 10 pieces, forming 40 thousandths. Again the 6 people each receive 6 thousandths with 4 thousandths left over. When one brownie is shared equally among 6 people, they each receive 0.166 brownie and there is 4 thousandths of the brownie left over. Notice that if we keep cutting the leftovers, the process will never end. Repeating decimals result when there are leftovers from the dividing process no matter how many times we divide.

When 17 brownies are shared equally among 8 people, each person gets 2.125 brownies. This decimal does not repeat but terminates, since there are no leftovers from the cutting process. ▲

Teaching Number Sense

Promoting the development of children's number sense is a complex and multifaceted task. Students' early understanding of number involves making sense of counting, decomposition, and place value. These ideas are the foundation of mathematics.

As students progress through school they build on the additive nature of number and consider multiplicative relationships involving factors, products, divisors, and multiples. Students in middle school expand their use of number to different sets of numbers such as rational numbers and irrational numbers. They must be aware of both the similarities and differences among sets of numbers.

Teachers play an important role in helping students develop number sense. Tasks that highlight relationships and properties as well as that focus on skills are essential. Reexamine your curriculum materials in light of some of the discussions in this chapter. Decide how you might incorporate some of these ideas into appropriate instructional tasks for your students.

Questions for Discussion

1. Numbers are classified in many ways. Which classification system do you find most useful? Why?
2. Why is the concept of evenness and oddness so important? How might a fourth grader explain that the sum of two odd numbers must be an even number?
3. How does a young child's concept of quantity develop? Discuss activities that will promote students' understanding and help them reach important milestones.
4. Numbers can be decomposed both additively and multiplicatively. What does it mean to understand number concepts from these two perspectives?
5. What features of signed numbers are essential for students to explore?
6. Using what you know about our base ten system, explain why the decimal expansion of $\frac{1}{12}$ is $0.08\overline{3}$. (See Chapter 6, page 133.)