CHAPTER 1

Patterns and T-Charts

Introductory Experiences

Over the years we have refined the ways we help our middle school students develop proficiency in finding rules for patterns. We have found that the approach that works best for us incorporates a wide variety of content and pedagogical focuses, including the following.

Content Focuses

- Exposing students to a variety of patterns over time
- Using relevant/real-world patterns early and often
- Working with students to recognize and describe, extend, and generalize each pattern whenever possible
- Starting with patterns created using concrete objects and encouraging use of drawings, words, and symbols as individual students are ready
- Beginning with simple, unambiguous patterns but progressing rapidly to more complicated ones
- Spiraling use of vocabulary such as *stage, constant, variable, iterative rule,* and *explicit rule*—students refine definitions over time
- Encouraging multiple interpretations of each pattern whenever appropriate
- Incorporating the use of a T-chart/table for each pattern
- Validating correct iterative rules while encouraging searches for appropriate explicit rules
- Asking students to use their rule(s) to predict what each pattern will look like or what its value will be for *several stages*
- Asking students to use their rule(s) to predict what each pattern will look like or what its value will be for *some large stage number*
- Asking students to use their rule(s) to predict what each pattern will look like or what its value will be for *any stage number*
- Asking students to find the appropriate stage number for a description or value of a particular stage of a pattern

- Moving as quickly as possible to working almost exclusively with patterns embedded in problem situations

Pedagogical Focuses

- Exploring one pattern at a time, followed by students' written reflections
- Working in a whole-class setting with built-in "think time" for the first few patterns—allowing students to work alone, then share their ideas with the class
- Working with individual students who need extra practice as the need arises
- Working in small groups and sharing ideas with the rest of the class as soon as appropriate in a particular class
- Having students write their reflections about a pattern early and often

The following vignette is from a lesson we used with a class of sixth graders, some of whom had very little experience working with patterns. We presented a simple pattern every few days early in the new school year. On each day following the exploration of a particular pattern, the students wrote brief reflections about the pattern they encountered the day before.

Exploration: The Rocket Pattern

Each student had a pencil and paper and was given a small bag of pattern blocks. I asked the students how many of them had worked with finding rules for patterns in their math classes in earlier grades, and nearly all raised their hands. I explained that the class would examine several patterns in the next few days and would be (1) sharing ways of thinking about various patterns, (2) practicing using a T-chart, and (3) hopefully, adding techniques for finding rules for patterns to each student's mathematical toolbox. I then used overhead pattern blocks to build the first three stages of the Rocket Pattern and projected them on the screen at the front of the room (see Figure 1–1).

I asked that each student look carefully at the first three stages of the pattern as they appeared on the screen and then build what he or she thought a Stage 4 rocket would look like.

Barry immediately raised his hand and asked, "What's a Stage Four rocket?"

There are usually sixth graders in any class who are not familiar with or do not remember the terms *stage* and *stage number*. Since it is essential that each student understand terms like these to be successful in our study of patterns, it is important to go over those terms with each new pattern until all the students are comfortable with sensible working definitions.

I asked whether anyone would like to say what he or she thought I meant, and Jack volunteered, "A Stage One, a Stage Two, and a Stage Three rocket are on the screen. They are the first three steps, or parts, in a pattern. If you look at how the pattern is going, you can tell what the next step in the pattern—a Stage Four rocket—will look like, and that's what you build with your pattern blocks."

Barry indicated that he understood Jack's explanation and each student successfully produced a Stage 4 rocket, as shown in Kenya's drawing in Figure 1–2.

I then asked that each student build, draw, or describe in words a Stage 5 rocket. Most of the students chose to build or draw a Stage 5 rocket. All except one of these students correctly depicted the figure. Sean had placed four blue rhombi under his three-piece model of the rocket. When I leaned over his shoulder and asked Sean to explain his thinking, he quickly exclaimed, "Oh, I meant to put five blue pieces there!" as he added a fifth rhombus to his model.

A few students chose to describe the Stage 5 rocket in words. Cemmie wrote, *A Stage 5 Rocket is just like all the others except it has 5 puffs of smoke.*

When asked to explain her thinking, Cemmie replied, "A Stage One rocket [pointing to the appropriate image on the screen] has a rocket made from three pieces plus the one puff of smoke. A Stage Two rocket has a rocket and two puffs of smoke [again pointing to the screen], and so on, so a Stage Five rocket has five puffs of smoke." When asked whether they agreed with Cemmie's line of reasoning, all the students nodded.

"So who can describe a Stage Ten rocket?" I asked. Several hands went up. I paused to give other students more time to think and about three-fourths of the students soon had their hands in the air.

I called on Suzi, who said, "It would be a rocket with ten puffs of smoke."

I asked whether everyone agreed and Camille spoke up. "Well," she said, "I would say it is a rocket made from a square, a triangle, and a trapezoid, and ten rhombuses for the puffs of smoke." Everyone agreed that

FIGURE 1–1

The rocket pattern

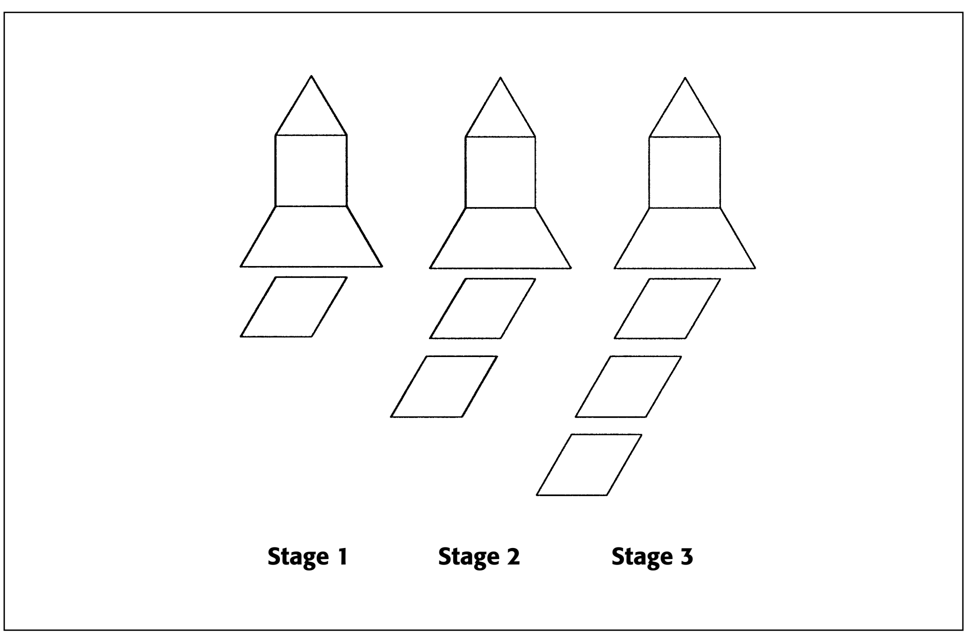
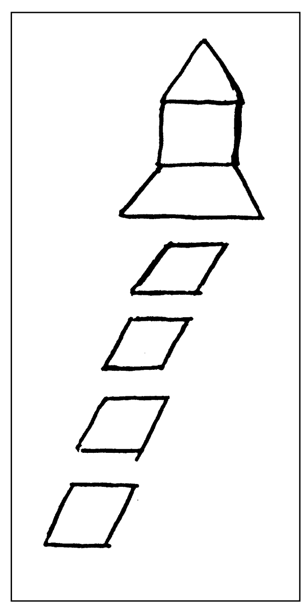


FIGURE 1–2

Kenya's drawing of a Stage 4 rocket



Camille's description was correct and also made it possible for someone who could not see a model or drawing to visualize the figure more clearly in his or her head.

I asked why some people had not raised their hands earlier, and Robbie volunteered that he got the same answer as Suzi and Camille but was not quite sure enough to raise his hand. I asked what he might do to be sure he was right. He replied, "Build or draw each one up to Stage Ten."

When asked, "Can you think of another way?" Robbie had no answer, so I asked him to choose another student to offer a suggestion. He called on Chelsie.

"Draw a T-chart," she offered. When Robble indicated that he had never seen a T-chart, I asked Chelsie to draw a T-chart for the Rocket Pattern on the board and explain it to the class. Figure 1–3 shows Chelsie's chart and explanation.

I still find that many students come to middle school claiming never to have heard of or seen a T-chart. Unfortunately, it is not simply that the term *T-chart* is unfamiliar: these students either have never been asked to record and examine data in a table, or they don't remember having done so. Either way, I make sure that early in the school year each of my classes is introduced to or reminded of using tables to record and interpret data. Through the years I have become convinced that using tables is the single best path to developing understanding of variables, a vital concept in the development of algebraic thinking. I want every student to develop facility with this powerful problem-solving tool during the middle school years.

To check his understanding, I asked Robbie to tell, in his own words, what Chelsie had said. He explained, "A T-chart is a way to write down and keep up with what is happening in a pattern. It can help you realize what is happening or, like for me, help you to know you are thinking right.

FIGURE 1–3 Chelsie's chart and explanation

Stage	Total # of Pieces
1	4
2	5
3	6
4	7
5	8
6	9
7	10
8	11
9	12
10	13

Chelsie's Comments

A T-chart has at least two parts, one column for the stage numbers and one for the numbers that match up with each of the stage numbers. For the Rocket Pattern—and a lot of other patterns—the second column is the total number of pieces or parts you get for that stage. You can count them from drawings, but soon you notice that the total number of pieces gets bigger by one each time, so you just fill in the rest of the chart.

For the Rocket Pattern you start with four pieces and add one more for each stage, so a Stage Ten Rocket has a total of thirteen pieces."

Emily raised her hand. "It makes sense to me that the total number of pieces increases by one for each stage because we added one puff of smoke each time. But if I think about it without looking at the chart, I get confused. We started with four pieces, so why isn't the total for a Stage Ten rocket fourteen pieces—four to start with and then ten puffs?"

Scot tried to explain. He pointed out, "Actually, Emily, you had only three pieces to start with—the triangle, the square, and the trapezoid (or the rocket). And then you added one rhombus (or the puff of smoke) to get a whole Stage One rocket."

Emily still looked confused, so I used her question as an opportunity to show the class how to expand the use (and often the value) of a T-chart. I pointed out that many people find it helpful to add a column to their T-chart, and I projected the image shown in Figure 1–4 onto the screen.

Next I asked Suzi to reread her description of a Stage 10 rocket. When she finished reading, I asked the class, "So what did Suzi see as the parts of each stage?"

Jared answered, "She said each stage has a rocket and some puffs of smoke."

I wrote the words *Rocket Pieces* and *Puffs* in the center column, then asked, "And what do you do to get the total number of pieces needed for any stage?"

"Add," the students said in unison, so I put a plus sign in the center column between the two phrases, as shown in Figure 1–5.

Next I asked whether anyone could look at the Stage 1 rocket, break it into the parts listed in the center column, and tell what numbers to write for Stage 1. Robin quickly waved her hand in the air and waited patiently while I gave the other students some time to try to find the an-

FIGURE 1–4 Blank T-chart for the rocket pattern

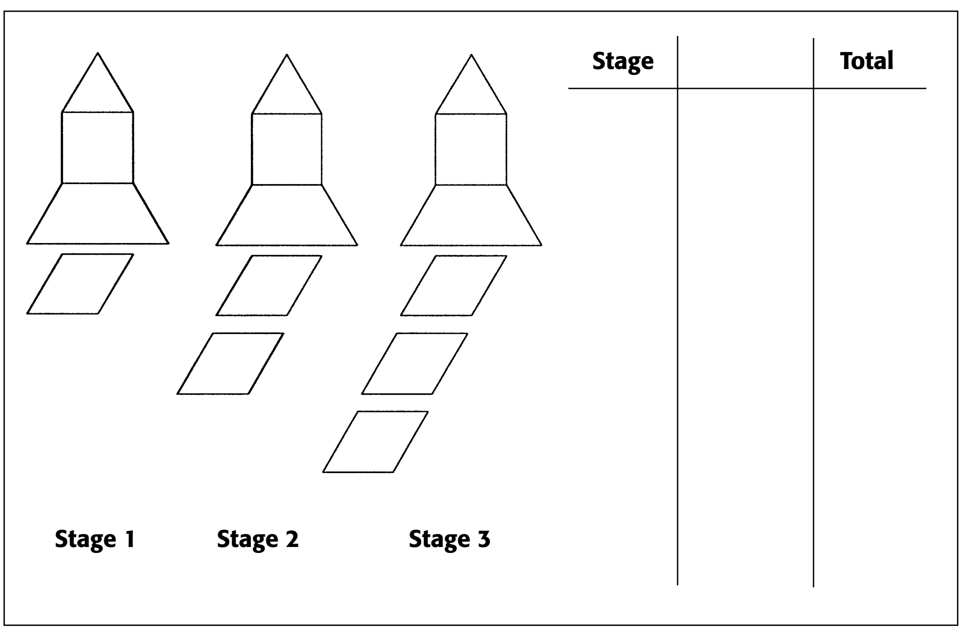
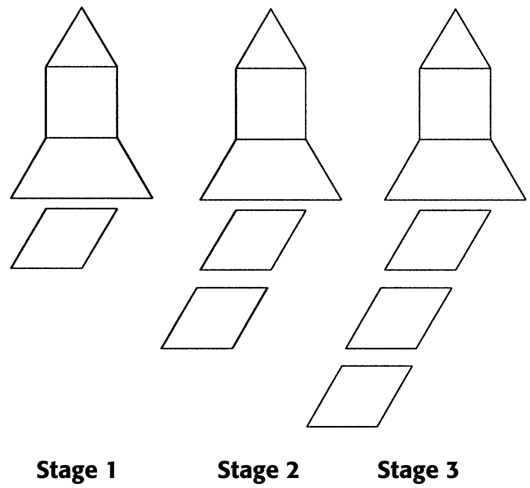


FIGURE 1–5 Adding to the T-chart for the rocket pattern



Stage	Rocket Pieces + Puffs	Total

swer for themselves. I then asked Robin to come to the overhead, explain her thinking, and write the appropriate numbers in the chart. She said the following while she filled in the chart, as shown here. “For the Stage One rocket there are three pattern blocks, or pieces, in the rocket itself. Also, there is one pattern block for the puff of smoke, so there is a total of four pieces.”

Stage	Rocket Pieces + Puffs	Total
1	**3 + 1**	**4**

Teddy raised his hand to comment, “I think we need to call the last column ‘Total Number of Pieces’ because all the pieces are not the same size or shape and that might confuse someone.” Several students nodded and Teddy rewrote the caption as he filled in the numbers for Stage 2 in the T-chart, saying, “For Stage Two, the rocket has the same three pieces, but there are two puffs of smoke, so the total is five pieces.”

Stage	Rocket Pieces + Puffs	Total # of Pieces
1	3 + 1	4
2	**3 + 2**	**5**

Emily agreed to do Stage 3. She smiled as she wrote the numbers and explained, “For Stage Three, you add three pieces for the rocket and three puffs of smoke, making a total of six pieces.”

Stage	Rocket Pieces + Puffs	Total # of Pieces
1	3 + 1	4
2	3 + 2	5
3	**3 + 3**	**6**

Next I asked the students to look carefully at the center column of our T-chart, try to find any overall pattern there, and raise their hands when they found such a pattern. Soon nearly every hand was raised. I then asked whether anyone could express the pattern he or she found in words. Daniel responded, "For every stage, there are three pieces in the rocket and one more puff than in the last stage." I asked for a show of hands by students who agreed and every student so indicated. I was surprised that no one first isolated the pattern of threes or the increasing pattern 1, 2, 3 . . . , but was thankful they seemed to grasp the overall pattern at once.

I asked whether anyone saw the pattern in a different way, but no one volunteered. I was hoping someone would offer the explicit pattern "For each stage there are three pieces in the rocket plus a number of smoke puffs that is the same as the stage number," but I decided not to push for this observation on the first day of our pattern explorations. Most middle school students come to us thinking iteratively—basing their understanding of a pattern on the change from one stage to the next. I think it is important to validate this way of thinking and not to rush students to find explicit patterns and rules, so I decided to revisit this pattern after a few more experiences and see whether I could push the students further at that time.

During this part of the exploration, I asked two more questions to introduce the ideas of constant and variable. First, I asked, "Who can tell us which part of the Rocket Pattern stays the same?"

Several students quickly answered, "The threes," or "The three pieces of the rocket," or simply, "The rocket." I affirmed their answers and explained that mathematicians refer to a part of a pattern that remains the same for every stage as a *constant*. In this way, I introduce essential vocabulary as it applies in a relevant context (in this case, the Rocket Pattern). I have found that this increases the students' initial understanding of vocabulary and makes it more likely that they will recall the meaning of such terms later. Marilyn announced that *constant* was a "sensible name" since her mother often used the word when Marilyn did something again and again, like "tease my brother *constantly*." I suspect that this comment will probably serve as a mnemonic device that will help some of the students remember the term better than the three pieces of the rocket in this pattern!

The second, related question I asked was, "Who can tell us which part of the pattern changes from stage to stage?" All agreed it was the number of puffs of smoke that changed, so I explained that mathematicians call any part of a pattern that changes *variable*. At this point, I did not expect the students to remember precise definitions but hoped instead that they would begin to develop an intuitive understanding of these important ideas.

Next I asked that each student try to think of a rule for the total number of pattern block pieces for any stage in the Rocket Pattern and write it down. I explained that someone should be able to read the rule and then find the total pieces needed for any stage of the Rocket Pattern. When everyone had something written down, I asked the students to share their rules. I also told them they would be able to write a revised version of their rule if they wished after hearing all the ideas. Jeremy's rule was "To get the total for any stage, add one to the total for the stage before it."

I asked whether this rule would work and most of the students nodded. I asked for a volunteer to use the rule for Stage 11. Teddy volunteered. "For a Stage Eleven rocket there would be fourteen pieces because there were thirteen pieces in Stage Ten and thirteen plus one equals fourteen." His classmates agreed.

I asked whether anyone else had written the same rule as Jeremy and over half the students raised their hands. Because this was our first exploration, I asked that each student read his or her rule aloud if it was different in any way—even if it was just one word—from Jeremy's rule. Many, like Sarah, indicated that they wanted to modify their rules slightly after hearing Jeremy's. She said, "I wrote, *Total equals add one*, but I need to add 'to the stage before' for my rule to be clear." I agreed with Sarah's conclusion.

Next I asked whether anyone had a different rule. Camille volunteered, "Total pieces equals three, plus one more than the rhombuses in the stage before." When asked to do so, Cemmie used Camille's rule for Stage 11 by adding three plus *eleven*, the number of rhombuses or puffs from Stage 10, plus one. Most of the remaining students basically expressed their rule like Cemmie.

I then asked whether the two rules were the same. Josh volunteered, "They are not exactly the same, but they both work. And for both rules, you have to know something about the stage before it to get the answer."

I explained to the class that Josh's comment had two important mathematical ideas we should discuss. First, as Josh pointed out, while the rules were not worded exactly the same, they both worked for the Rocket Pattern. I told the class that when this is true, the rules are called *equivalent*. I explained that there are often several equivalent rules for a pattern—that might look or sound very different—all of which give the same results and all of which are correct. Again, I wanted to introduce the students to an idea that would be very important in our exploration of patterns.

Second, I explained that the rules were alike in another important way: both rules depended on knowing the results of the previous stage. I told the students that such rules are called *iterative* rules. I elaborated that such rules can be used to describe many real-world situations and are used in computer spreadsheets. I think it is important that students realize the value of iterative rules as well as their limitations.

When I asked whether anyone had another rule to share, Thomas raised his hand. "My rule is 'three plus the number of puffs, which is the same as the stage number,' or three plus *s*."

There were, as I anticipated, more than a few frowns around the room. I asked Thomas to explain his thinking and use his rule for Stage 11. Thomas put the transparency the students had constructed earlier back on the overhead (shown below).

Stage	**Rocket Pieces + Puffs**	**Total # of Pieces**
1	3 + 1	4
2	3 + 2	5
3	**3 + 3**	**6**

He said, "For every stage, you have the three pieces in the rocket [pointing to the 3s in the column under "Rocket Pieces"]. And for every stage you have some puffs of smoke [pointing to the column of numbers under "Puffs"]. But I noticed that the number of puffs is always the same as the stage number; like for Stage One, there is one puff, for Stage Two, there are two puffs, and so on. So, for any stage, you can always add the three pieces of the rocket and the number of puffs, which you know matches the stage number. For Stage Eleven, you would add three plus eleven equals fourteen."

Several of the other students immediately seemed to understand Thomas's reasoning and its implications. "With Thomas's rule," Chelsie blurted, "I can do *any* stage quickly. That's cool."

"I don't get it," Barry said. "Can you show me the rule again? And what is the *s*?"

I interrupted to comment on Barry's questions. "In our class," I explained, "it is one of every student's jobs to ask questions when you don't understand. We want a classroom where we all work together to share ideas and understand as many mathematical ideas as possible. This is not possible without your asking questions. We won't all think the same way, and we won't all understand ideas at the same time or in the same way, but sharing ideas and asking questions will help us all to grow and to enjoy doing mathematics." During the first part of the school year I use every opportunity I can find to help foster the learning environment I hope to establish in that class. Meanwhile, I added, "Thomas, can you explain your rule again?"

This time, Thomas went to the overhead projector. He added a row in the T-chart for Stage 20 as he spoke.

Stage	Rocket Pieces + Puffs	Total # of Pieces
1	3 + 1	4
2	3 + 2	5
3	3 + 3	6
20	**3 + 20**	**23**

"My rule is 'Total equals three plus *s*.' For Stage Twenty, you would add three, the number of pieces in the rocket, plus twenty, the number of puffs of smoke. You know there are twenty puffs because the number of puffs is always the same as the stage number [pointing back and forth to the columns under "Stage" and "Puffs"]. Three for the rocket plus twenty puffs equals twenty-three total pieces. The *s* just stands for 'stage number.' I learned you can do that last year."

Barry nodded slowly. "I think I understand what you are saying. I might be able to use your rule, but I would never have thought of it."

I assured Barry that, soon, every student in the class would be discovering rules like Thomas's. I asked everyone to try using Thomas's rule for Stage 100. I suggested that each student try to picture the rocket in his or her head and look back at the table on the screen before he or she started using the rule. I then asked Barry to try to explain what he did. He went to the overhead and said, "For Stage One Hundred [writing *100*], there would be three pieces in the rocket [writing the *3*] plus [writing the + sign], mmmm, one hundred for the puffs of smoke [writing *100*], for a total of one hundred three pieces [writing the sum]."

Stage	Rocket Pieces + Puffs	Total # of Pieces
1	3 + 1	4
2	3 + 2	5
3	3 + 3	6
20	3 + 20	23
100	**3 + 100**	**103**

"Did everyone get that answer?" I asked. Suzi and three other students indicated that they did not write anything, but followed Barry's explanation and now thought they understood. Based on that input, I had the students try using $T = 3 + s$ for Stage 50, without writing in the T-chart if they could. All reported success.

"Thomas's rule," I explained, "is called an *explicit* rule, because you do not have to know the answer to the previous step in order to use the rule and get an answer for any stage. The *s* in the rule is called a *variable*, because it stands for a value in the pattern that changes from stage to stage. In this rule, *s* stands for the stage number, or the number of puffs, and *s* equals twenty for Stage Twenty while *s* equals one hundred for Stage One Hundred. Does that make sense to you?"

The students nodded and I did not question them further, although I knew it was most likely that some had only a glimmer of what I had just said. After all, this was our very first exploration! Understanding of variables does not come instantly for most students, but, over time, with many varied experiences, every student can develop meaning for this and other important mathematical concepts that comprise algebraic thinking.

I concluded by saying, "I want you to think about the Rocket Pattern and what you learned today overnight. Tomorrow I am going to ask you to write a short paragraph about our exploration. Are there any questions?"

Sarah asked, "Will we have to write about *everything* we did?"

I laughed and assured Sarah and the rest of the class that they would simply explain the Rocket Pattern and a rule for finding the total number of pieces needed to build any stage.

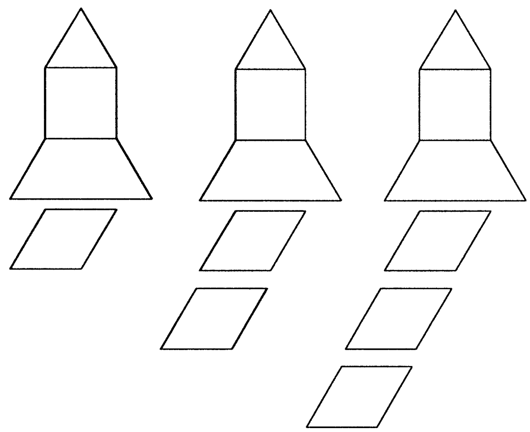
As expected, it took quite a while for the class to complete this first pattern exploration. For most students in the class, only the first layer of understanding about finding rules for patterns was developed. Also, important concepts such as variable were only introduced or were addressed briefly, while others did not emerge at all. However, I am confident that working slowly through the first few explorations and discussing concepts and vocabulary as they arise naturally during those explorations is best. This approach results in a stronger foundation and more long-term success than trying to cover a larger number of examples quickly or simply presenting concepts in an order that seems logical to me. I know the content and pedagogical objectives I hope to accomplish by the end of a number of explorations and have confidence, based on past experience, that these objectives will be met for almost all the students by the end of several explorations. Thus, I ended the class feeling optimistic and looking forward to reading the students' reflections about their experience with the Rocket Pattern.

I started class the next day by placing the transparency shown in Figure 1–6 on the overhead projector.

I also explained, "You can write out the method you used yesterday to describe the Rocket Pattern and how it grows, a different method, or more

FIGURE 1–6 Launch for reflections about the rocket pattern

Yesterday we explored the Rocket Pattern shown below. Explain in writing a rule for what the pattern looks like, how it grows, and how many total pieces are needed to build it for any stage.



Stage 1 **Stage 2** **Stage 3**

than one method. I want to know your ideas and how you think about this pattern."

The students took a few minutes to complete their reflections and then turned in their papers. We spent the rest of class on another topic and I read their papers that night in preparation for our second patterns exploration the following day.

Emily's paper was typical of most of the students' reflections (see Figure 1–7).

Emily's writing showed that she could describe the pattern adequately and provide an appropriate iterative rule. Like most of the students in this class, her reflection was not very detailed. I have found that most students need to hear several examples from their classmates that are more expansive and clearer than their own before they begin to include important details in their own reflections.

The writing of some students showed more understanding than that of their classmates. Kenya, for example, wrote the reflection in Figure 1–8 on page 14. A few students' reflections indicated some incomplete or inaccurate understandings (see Barry's writing in Figure 1–9 on page 14).

FIGURE 1–7 Emily's reflection

Each stage of the pattern includes a rocket with 3 pieces plus some puffs of smoke. Each stage has one more puff so there is one more total number of pieces.

FIGURE 1–8 Kenya's reflection

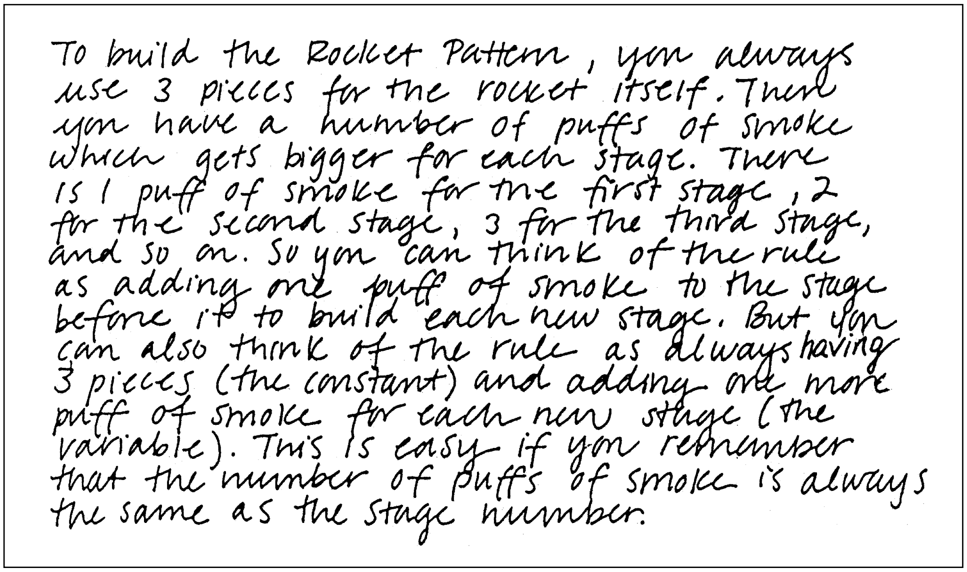
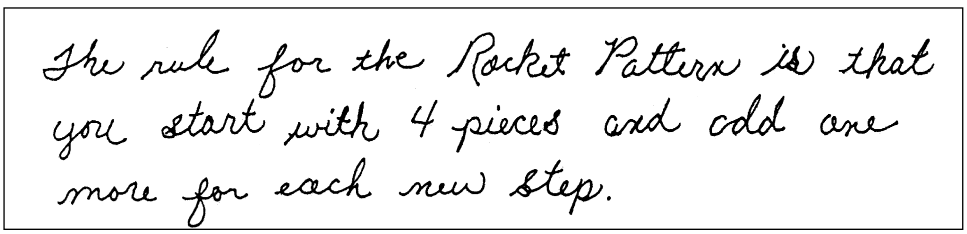

To build the Rocket Pattern, you always use 3 pieces for the rocket itself. Then you have a number of puffs of smoke which gets bigger for each stage. There is 1 puff of smoke for the first stage, 2 for the second stage, 3 for the third stage, and so on. So you can think of the rule as adding one puff of smoke to the stage before it to build each new stage. But you can also think of the rule as always having 3 pieces (the constant) and adding one more puff of smoke for each new stage (the variable). This is easy if you remember that the number of puffs of smoke is always the same as the stage number.

FIGURE 1–9 Barry's reflection


The rule for the Rocket Pattern is that you start with 4 pieces and add one more for each new step.

While Barry's iterative rule yields the correct total number of pieces for any stage, his writing shows no understanding of the constant and variable parts of the pattern. Of course, he might have just failed to write down this information—understanding it after all—but, with the most generous interpretation, his writing indicates that he did not think this information was important. His reflection indicates that Barry needed more experiences with finding rules for patterns.

Only Thomas included the explicit rule with a variable in his reflection.

The next day in class we had a brief discussion about the reflections. I reported that most members of the class wrote the rule as, "The total number of pieces for any stage equals the number of pieces in the previous stage plus one." I made a transparency of Kenya's reflection and had her read it to the class. The other students agreed that the details she included made her explanation about the pattern and the connection of the rule to the models or drawings clearer than simply stating the rule. They also agreed that Kenya's writing indicated understanding of the terms *constant* and *variable* as well. I acknowledged that I knew other students in the class understood these same ideas and connections, and encouraged all the students to include such information in their reflections in the future. I felt the class was ready for another exploration.

Follow-Up Explorations

Figures 1–10 and 1–11 show two additional patterns that I have used with middle school students on days soon after the student exploration of the Rocket Pattern. The iterative and explicit rules are given following an illustration of the first three stages of each pattern. Notice that each of these patterns is simple and unambiguous. I do not push the students to arrive at an explicit rule at this point.

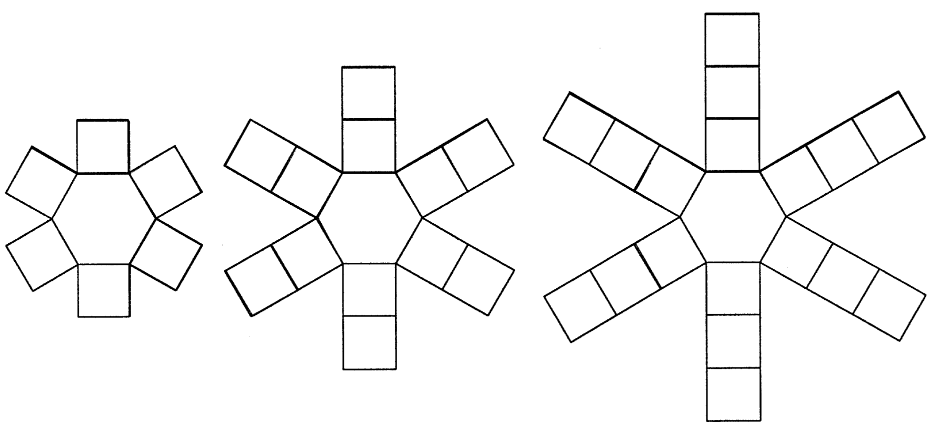
The following elements of the exploration with the Rocket Pattern should be repeated with patterns like these:

- Describing, building a model for, and/or drawing Stages 4 and 5 of the pattern
- Predicting, then verifying, what Stage 10 of the pattern would look like
- Using a T-chart to record information, search for patterns, and find and check a rule for the pattern
- Finding an iterative and explicit (*if* it comes from the students) rule for the pattern
- Using the rule(s) to find any stage of the pattern
- Writing and sharing reflections the following day

Another pattern that is appropriate to use with students at this beginning level of experience, titled "Exploring Houses," including a Blackline

FIGURE 1–10 Hexagonal sun pattern

Find a rule for the total number of shapes for any stage.



Stage 1 **Stage 2** **Stage 3**

Iterative Rule

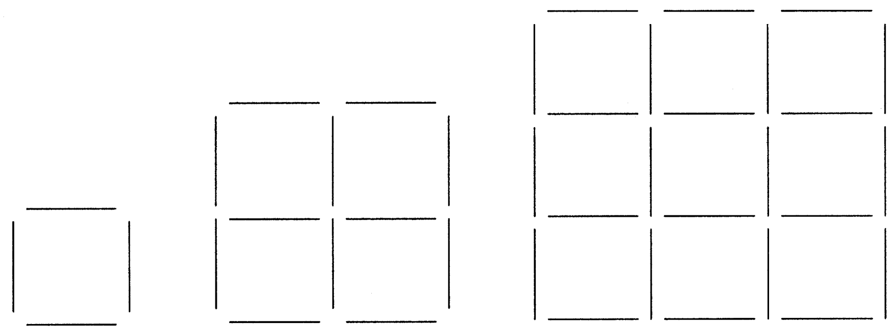
Add 6 to the total number of blocks in the previous stage.

Explicit Rule

Total = 6n + *1 when* n = *stage number.*

FIGURE 1–11 Toothpick perimeter

Find a rule for the total number of toothpicks in the perimeter for any stage.



Stage 1 **Stage 2** **Stage 3**

Iterative Rule

Add 4 to the total number of toothpicks in the perimeter of the previous stage.

Explicit Rule

Total = 4n *when* n = *stage number.*

Master and discussion, can be found in *Navigating Through Algebra in Grades 6–8* (Friel et al. 2001, 9, 74).

Often a class will need only one follow-up pattern before I feel comfortable proceeding to patterns that lend themselves to more than one interpretation. Occasionally, a class or group of students will need to work with several additional simple patterns before encountering less straightforward patterns.

When most of the students in a class appear comfortable with extending patterns, using a T-chart, and finding iterative rules for patterns, it is appropriate to use the lesson *Surprising Squares*, which emphasizes finding explicit rules for patterns.